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This report describes progress in the development and application of H-infinity-optimal control theory to distributed parameter systems. This research is intended to develop both theory and algorithms capable of providing realistic control systems for physical plants which are appropriately modeled as infinite dimensional linear systems, such as large space structures. In pursuing this program, we have focused on issues motivated by specific models of infinite dimensional systems.

Our main results are as follows: We have extended outer factor absorption results to cover certain irrational outer factors. This is in order to justify transformations used in a step for further explicit solution. We have generalized previous results on the single-input/single-output (SISO) mixed sensitivity problem to the case of irrational outer factor and unstable plant. We have solved a multi-input/multi-output mixed sensitivity problem which cannot be treated by previous results. Finally, we developed a technique for numerically computing the optimal value of weighted mixed sensitivity for SISO systems, for the case where explicit symbolic inner/outer factorizations are not possible.

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H^∞-Optimal Control for Distributed Parameter Systems

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Chapter 1

Overview

1.1 Introduction

The research described in this report concerns the following fundamental issues in control system design:

1. Selection of meaningful yet analytically tractable performance criteria,

2. Sensitivity of closed loop stability and performance to both structured and unstructured plant uncertainty and model errors, and

3. Approximation of ideal compensators with implementable ones.

Since the seminal work in this area [52], there has been very substantial progress. [21] contains a good survey and a comprehensive bibliography for the field up to the time of writing in 1986. Since then developments for linear lumped parameter systems have focussed on computational techniques and applications. Although [52] suggests applicability to systems with irrational transfer functions, actual development of a theory taking into account the characteristics of infinite dimensional plants began with the mutually independent work [9] and [19].

During the last five years there has been significant progress on results for distributed parameter systems. However, the progress has not matched that of the results lumped parameter systems over a comparable period. This is partly because the research related to $\mathbf{H}^\infty$ control of distributed parameter systems has generally appeared in a very abstract, highly mathematical form, in which the connection to real problems is made only in a much-used simple delay example.

In the work we describe here, we have focussed our theoretical work on issues which arise in applying the theory to non-trivial examples. These examples can serve as archetypes of distributed parameter systems of interest to the Air Force.

This research is intended to develop both theory and algorithms capable of providing realistic control systems for physical plants which are appropriately modeled as infinite dimensional linear systems. Typical physical plants would be large orbiting radar antennas,

large robot arms such as the shuttle Remote Manipulator System, and high precision orbiting optical systems such as surveillance satellites. We hypothesize that maintaining an infinite dimensional model as long as possible during the design process will lead to designs which better capture intrinsic infinite dimensional characteristics, at lower computational cost and with greater analytical certainty, than alternative procedures which employ finite dimensional approximations as a precursor to design. However, we must ultimately check this hypothesis by comparing designs.

In pursuing this program, our approach is twofold: First, we must examine sufficiently realistic models of typical prototype plants to see what characteristics models for theoretical analysis should have in order to yield useful results. Our work to date along this line indicates that much existing theoretical work makes overly strong assumptions, which prevent the work from being applicable to even basic plants of interest. Having distilled from such prototype plants what we consider to be essential characteristics, we must derive theoretical procedures for obtaining a solution.

Secondly, we recognize two facts about infinite dimensional models, and direct our research to deal with these realities: One fact is that even infinite dimensional models will themselves only be approximations, and the other is that infinite dimensional models which have the fidelity we seek to any complex physical system will probably not be tractable analytically (in the sense of our being able to compute explicit expressions as we can for simple delay systems). Thus, having an explicit inner/outer factorization as we allow ourselves to assume in order to develop theoretical results, is not a realistic assumption, even for something so relatively simple as a damped Timoshenko beam model (compared to the shuttle Remote Manipulator System).

1.2 Summary

Our main results are as follows:

1. (Chapter 2) We have also calculated and analyzed transfer functions for certain beam models [11], [31], for use as testbeds of our theoretical results. We now have five transfer function models of differing complexity, derived from physically motivated plants, which have led us to the different issues we are studying. These are: the delay (with rational factor), the damped Euler-Bernoulli beam, the damped Timoshenko beam, the two-delay multivariable plant (with rational factors), and the multivariable damped Euler-Bernoulli beam with torsion.

2. (Chapter 3) We have extended outer factor absorption results to cover certain irrational outer factors [12]. This ensures that this critical step (which is frequently ignored in the literature) has a sound basis. The method of proof may yield a useful computational technique for implementation, as well.

3. (Chapter 4) We have solved the scalar mixed sensitivity problem with irrational outer factors and rational weighting functions [13], [14]. In [12] we explicitly calculated a standard transformation of this criterion, leading us to the observation that in the case of plants with poles in the closed left half plane, for independently determined weighting functions the infimal norm is independent of the outer factor of the plant. On the other hand, we also

observe that the choice of weighting functions should depend on the outer factor of the plant, and in fact this reasoning leads to a need to extend results further to the case of irrational complementary sensitivity weighting.
4. (Chapter 5) We have successfully solved a mixed sensitivity problem for a non-trivial multi-input/multi-output system [47]. We believe this is the first such example solved. (In [34] results are presented which appear quite general, but in fact implicitly assume the satisfaction of a certain special commutativity property which does not usually hold, as in our example.)
5. (Chapter 6) We have developed a technique for numerically computing the infimal norm of the mixed sensitivity criterion without an explicit inner/outer factorization of the plant [16].

1.3 Papers and Reports

The following is a list of publications written using the support of this contract. The results in these documents have been incorporated into the present report.

1. D. S. Flamm, "A Model of a Damped Flexible Beam," ISS Report No. 54, June 14, 1990, Department of Electrical Engineering, Princeton University.

2. D. S. Flamm, "Outer Factor 'Absorption' for $\mathbf{H}^\infty$ Control Problems," ISS Report No. 55, July 31, 1990, Department of Electrical Engineering, Princeton University. (Submitted to *The International Journal of Robust and Nonlinear Control.*)

3. D. S. Flamm and H. Yang, "Some Comments on the $\mathbf{H}^\infty$-Optimal Scalar Mixed Sensitivity Problem," ISS Report No. 56, August 16, 1990, Department of Electrical Engineering, Princeton University.

4. D. S. Flamm and H. Yang, "$\mathbf{H}^\infty$-Optimal Mixed Sensitivity for General Distributed Plants," ISS Report No. 57, August 31, 1990, Department of Electrical Engineering, Princeton University. (Submitted to *IEEE Trans. Automatic Control.*)

5. D. S. Flamm and H. Yang, "$\mathbf{H}^\infty$-Optimal Mixed Sensitivity for General Distributed Plants," *Proc. 1990 Conference on Decision and Control*, Dec. 5-7, 1990, pp. 134-139, Honolulu, HI. (condensed version of ISS Reports 56 and 57.)

6. D. S. Flamm, H. Yang, Q. Ren and K. Klipec, "Numerical Computation of Inner Factors," ISS Report No. 58, August 8, 1990, Department of Electrical Engineering, Princeton University.

7. H. Yang and D.S. Flamm, "Mixed Sensitivity Design for a Real Multivariable Delay Problem," ISS Report No. 60, Sept. 7, 1990, Department of Electrical Engineering, Princeton University. To appear in *Proc. American Control Conference*, June 26-28, 1991, Boston, MA.

1.4 Notation

$\mathbf{H}^2$, $\mathbf{H}^\infty$: Hardy spaces on the right half plane.

$\mathbf{L}^1$, $\mathbf{L}^2$, $\mathbf{L}^\infty$: Lebesgue spaces on the real axis $\mathbf{R}$ or imaginary axis $j\mathbf{R}$.

$\mathbf{H}^2_-$: the orthogonal complement of $\mathbf{H}^2$ in $\mathbf{L}^2$, so $\mathbf{L}^2 = \mathbf{H}^2 \oplus \mathbf{H}^2_-$.

$\mathbf{L}^\infty_{2\times 1}$, $(\mathbf{H}^2_{2\times 1})_-$, *etc.*: matrices with entries in the corresponding spaces.

C: the subspace of $\mathbf{L}^\infty$ consisting of bounded continuous functions.

Π_+: the projection $\mathbf{L}^2 \to \mathbf{H}^2$.

Π_-: the projection $\mathbf{L}^2 \to \mathbf{H}^2_-$.

$\Pi_\mathbf{K}$: the projection $\mathbf{L}^2 \to \mathbf{K}$, where $\mathbf{K}$ is a closed subspace.

$\Pi_\mathbf{K}$: the projection from $\mathbf{L}^2$ onto a subspace $\mathbf{K}$.

$x^*(s)$: the involution of $x(s)$, *i.e.*, $x^*(s) \triangleq \bar{x}(-\bar{s})$. Here the bar denotes the complex conjugate.

$x^{(*)}(s) \triangleq x^*(s)x(s)$.

x_i, x_o: inner and outer factors, respectively, of $x \in \mathbf{H}^\infty$.

Chapter 2

Models

In this chapter we present four models which we have been examining in order to motivate and guide our investigation.

Since one of the main applications in which we are interested is the control of large space structures, three of the models are of flexible beams. We think of these as building blocks from which we well construct more complex and more realistic space structure models as the theory of design with them matures. The simplest model is a Euler-Bernoulli beam with viscous damping. We also present here a damped Timoshenko beam model, and beam model with two degrees of freedom. The Euler-Bernoulli beam is a single-input/single-output model which we have analyzed in substantial detail, and its characteristics have influenced several of our directions of research. The Timoshenko beam is also a SISO model, but it is substantially more difficult to analyze, mainly because an explicit expression for a transfer function does not seem to be available. We present it here to illustrate the complexity involved, and as an object of future study. Similarly, the transfer funcion for the two-degree-of-freedom beam is intractable analytically, but we intend it as an initial object of future study for multi-input/multi-output space structure models.

The fourth model is a multivariable delay model for an automotive system. This is of interest for several reasons: it is a realistic MIMO irrational transfer function, it was an intractable problem in its exact form (prior to the current work), and it has been examined by means of approximation by other researchers, so that computational results are available for comparision purposes.

2.1 A Damped Euler-Bernoulli Beam

The model we present in this section is relatively tractable by analytical means. We are able to find exact singularities and approximate zeros, and to characterize the asymptotic behavior for high frequency.

2.1.1 Model Derivation

With the addition of two tractable types of viscous damping to the classical Euler-Bernoulli beam model, one obtains for the partial differential equation [Clough and Penzien, p. 302],

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2} + c_s I\frac{\partial^3 v}{\partial x^2 \partial t}\right) + m\frac{\partial^2 v}{\partial t^2} + c\frac{\partial v}{\partial t} = p \tag{2.1}$$

where $v(x,t)$ is the lateral displacement of the beam, $p(x,t)$ is the external load on the beam, c is the resistance to transverse velocity and c_s is the resistance to strain velocity. c has the interpretation of external damping, and c_s has the interpretation of internal damping. We are interested in the case of no external load, ($p(x,t) = 0$), and we assume a beam of length 1 with free boundary conditions at both ends.

With the intent of computing the transfer function from a torque applied at one end to the displacement at the other end, we take Laplace transforms, obtaining

$$(EI + c_s Is)v^{(4)}(x,s) + \left(ms^2 + cs\right)v(x,s) = 0\,. \tag{2.2}$$

The problem is separable, and setting

$$v(x,s) = \varphi(x)\cdot w(s) \tag{2.3}$$

we obtain

$$\frac{\varphi^{(4)}(x)}{\varphi(x)} = a^4 = -\frac{ms^2 + cs}{EI + c_s Is}\,. \tag{2.4}$$

The solution is of the form

$$v(x,s) = (k_1\cos(ax) + k_2\sin(ax) + k_3\cosh(ax) + k_4\sinh(ax))w(s) \tag{2.5}$$

where $w(s)$ and the constants k_i are determined by the boundary conditions. The boundary conditions at each end are related to the constants k_i by the equations

$$\begin{pmatrix} v(0,s) \\ v'(0,s) \\ v''(0,s) \\ v'''(0,s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & a & 0 & a \\ -a^2 & 0 & a^2 & 0 \\ 0 & -a^3 & 0 & a^3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} w(s) \tag{2.6}$$

and

$$\begin{pmatrix} v(L,s) \\ v'(L,s) \\ v''(L,s) \\ v'''(L,s) \end{pmatrix} = \begin{pmatrix} \cos(aL) & \sin(aL) & \cosh(aL) & \sinh(aL) \\ -a\sin(aL) & a\cos(aL) & a\sinh(aL) & a\cosh(aL) \\ -a^2\cos(aL) & -a^2\sin(aL) & a^2\cosh(aL) & a^2\sinh(aL) \\ a^3\sin(aL) & -a^3\cos(aL) & a^3\sinh(aL) & a^3\cosh(aL) \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} w(s) \tag{2.7}$$

The relation between the boundary conditions at either end can be expressed as

$$
\begin{pmatrix} v(L,s) \\ v'(L,s) \\ v''(L,s) \\ v'''(L,s) \end{pmatrix} = \frac{1}{2} \cdot
\begin{pmatrix}
\cos(aL) + \cosh(aL) & \frac{\sinh(aL)+\sin(aL)}{a} & \frac{\cosh(aL)-\cos(aL)}{a^2} & \frac{\sinh(aL)-\sin(aL)}{a^3} \\
a(\sinh(aL) - \sin(aL)) & \cos(aL) + \cosh(aL) & \frac{\sinh(aL)+\sin(aL)}{a} & \frac{\cosh(aL)-\cos(aL)}{a^2} \\
a^2(\cosh(aL) - \cos(aL)) & a(\sinh(aL) - \sin(aL)) & \cos(aL) + \cosh(aL) & \frac{\sinh(aL)+\sin(aL)}{a} \\
a^3(\sinh(aL) + \sin(aL)) & a^2(\cosh(aL) - \cos(aL)) & a(\sinh(aL) - \sin(aL)) & \cos(aL) + \cosh(aL)
\end{pmatrix}
\cdot \begin{pmatrix} v(0,s) \\ v'(0,s) \\ v''(0,s) \\ v'''(0,s) \end{pmatrix}
\tag{2.8}
$$

Free boundary conditons at both ends, plus a torque $T(s)$ at the $x = L$ end, result in

$$
\begin{cases} v''(0,s) = 0 \\ v'''(0,s) = 0 \\ EIv''(L,s) = T(s) \\ v'''(L,s) = 0 \end{cases} \tag{2.9}
$$

Solving these simultaneous equations for $v(L,s)$ and $T(s)$ gives us the transfer function from a torque $T(s)$ at the $x = L$ end to the displacement at $x = 0$. Taking $L = 1$ we obtain

$$
P(s) = \frac{v(0,s)}{T(s)} = \frac{\cos(a) - \cosh(a)}{EIa^2(\cos(a)\cosh(a) - 1)} \tag{2.10}
$$

where

$$
a^4 = -\frac{s^2 + \frac{c}{m}s}{\frac{EI}{m} + \frac{c_d I}{m}s} . \tag{2.11}
$$

2.1.2 Essential Singularities

The appearance is that the functions appearing in the numerator and denominator of (2.10) may have essential singularities both in the left half plane and on the imaginary axis, because the arguments to these functions are roots of a rational function of s.

We shall show that we can factor the transfer function as the quotient of two functions, each of which is the composition of a holomorphic function with a rational transformation of the complex plane. Still, numerator and denominator are not analytic functions of s. This rational transformation has a pole in the left half plane, and we shall show that the pole becomes an essential singularity. Although the essential singularities for both the numerator and denominator are at the same point in the complex plane there is not cancellation.

Using power series expansions we find that

$$
\cos(a) - \cosh(a) = -2\left(\frac{a^2}{2!} + \frac{a^6}{6!} + \frac{a^{10}}{10!} + \dots\right) \tag{2.12}
$$

and

$$\cos(a)\cosh(a) - 1 = -\frac{2^2}{4!}a^4 + \frac{2^4}{8!}a^8 - \frac{2^6}{12!}a^{12} + \dots \tag{2.13}$$

Thus

$$N(a) = \frac{\cos(a) - \cosh(a)}{a^2} \tag{2.14}$$

is a holomorphic function of a^4, and a^4 is in turn a rational function of s, as claimed. Similarly for $D(a) = EI(\cos(a)\cosh(a) - 1)$.

This allows us to express the transfer function in the form

$$P(s) = \frac{-2\left(\frac{1}{2!} + \frac{\left(\frac{s^2+\frac{c}{m}s}{\frac{EI}{m}+\frac{c_s I}{m}s}\right)}{6!} + \frac{\left(\frac{s^2+\frac{c}{m}s}{\frac{EI}{m}+\frac{c_s I}{m}s}\right)^2}{10!} + \dots\right)}{EI\left(\frac{s^2+\frac{c}{m}s}{\frac{EI}{m}+\frac{c_s I}{m}s}\right)\left(-\frac{2^2}{4!} + \frac{2^4}{8!}\left(\frac{s^2+\frac{c}{m}s}{\frac{EI}{m}+\frac{c_s I}{m}s}\right) - \frac{2^6}{12!}\left(\frac{s^2+\frac{c}{m}s}{\frac{EI}{m}+\frac{c_s I}{m}s}\right)^2 + \dots\right)} \tag{2.15}$$

Since numerator and denominator are holomorphic functions of a^4, an essential singularity could only occur at infinity, and it is easy to see that this in fact occurs. Furthermore, this essential singularity of numerator and denominator is not a removable singularity of the transfer function, *i.e.*, there is not cancellation between numerator and denominator. One can see this by observing that (2.10) has no limit as $a \to \infty$, say along the real axis. One can check that $a \not\to \infty$ for values of s in the right half plane: a^4 becomes unbounded only as $s \to \infty$, and, if $c_s \neq 0$, as $s \to -\frac{E}{c_s}$. These are the essential singularities of the transfer function.

Figure 2.1 shows the behavior of (2.10) as $s \to \pm\infty$ along the real axis.

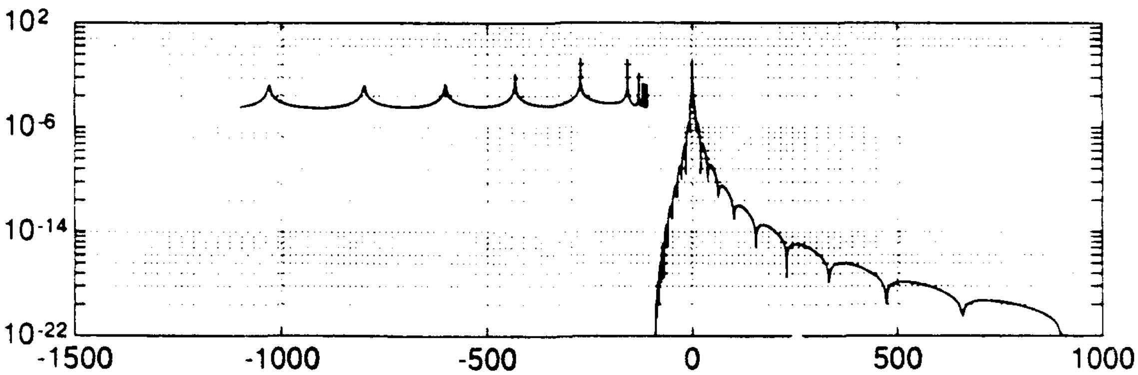

Figure 2.1: $|P(s)|$ as $s \to \infty$ along the real axis. ($m = 1$, $E = .1$, $I = .1$, $c = .001$, $c_s = .001$)

2.1.3 Analysis of Poles and Zeros

The analysis continues by finding the zeros and singularities of the numerator and denominator of the plant transfer function. For this purpose we shall first regard the transfer function

as a function of a. So we factor the beam transfer function as $P(s) = \frac{N(a(s))}{D(a(s))}$ with $N(a)$ and $D(a)$ as above.

The zeros of $N(a)$ are given by $a = (1 \pm j)n\pi$, for $n = 1, 2, \ldots$, or $a^4 = -4(n\pi)^4$. Notice that the zero corresponding to $n = 0$ is cancelled by the denominator in (2.14). Taking $\alpha_n = -4(n\pi)^4$, we note that the values of α_n are real and negative.

Let β_n be the values which a^4 assumes at the zeros of $D(a)$. We can show that the values β_n are all real and positive, and asymptotically approach the values $\left(\frac{(2n+1)\pi}{2}\right)^4$. Obviously $a^4 = 0$ is also a zero.

In terms of the Laplace transform variable s, using (2.11) we find that the numerator zeros are given by

$$\left.\begin{array}{r} z_{2n} \\ z_{2n+1} \end{array}\right\} = \frac{-(\alpha_n c_s I + c) \pm \sqrt{(\alpha_n c_s I + c)^2 - 4m\alpha_n EI}}{2m} \tag{2.16}$$

for $n = 1, 2, \ldots$.

The denominator is zero at $s = 0$, and at the points $s = -\frac{c}{m}$ and

$$\left.\begin{array}{r} p_{2n} \\ p_{2n+1} \end{array}\right\} = \frac{-(\beta_n c_s I + c) \pm \sqrt{(\beta_n c_s I + c)^2 - 4m\beta_n EI}}{2m} \qquad , n = 1, 2, \ldots \tag{2.17}$$

which are all in the left half plane. Both numerator and denominator are unbounded in every neighborhood of the point $s = -\frac{E}{c_s}$, and have zeros dense at that point. This is consistent with the fact that the point is an essential singularity of the transfer function as mentioned above. Figure 2.2 shows a typical pattern of poles and zeros of the transfer function, with $\frac{E}{c_s} = 100$.

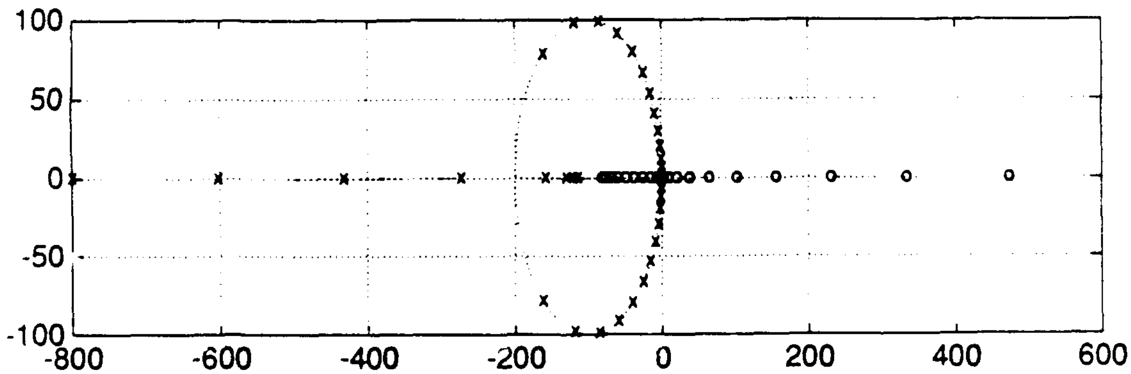

Figure 2.2: Poles and Zeros of Beam Transfer Function ($m = 1, E = .1, I = .1, c = .001, c_s = .001$)

2.1.4 Factorization of the Transfer Function

Since $\frac{s}{s+1}P(s) \in H^\infty \cap C$, the inner factor must be the product of a Blaschke product and possibly an exponential of the form $e^{-s\alpha}$ with $\alpha \geq 0$. But $e^{s\alpha}\frac{s}{s+1}P(s)$ is unbounded on

the positive real axis, so $\alpha = 0$. Thus the transfer function can be factored as $P(s) = P_u(s)P_i(s)P_o(s)$, where $P_u(s) = \frac{s+1}{s}$,

$$P_i(s) = \prod_{z_n \in \mathbf{C}^+} \frac{s - z_n}{s + z_n} \cdot \frac{|1 - z_n^2|}{1 - z_n^2} \tag{2.18}$$

and $P_o(s) \in \mathbf{L}^1 \cap \mathbf{C}$. P_o has essential singularities at the points $s = \infty$ and $s = -\frac{E}{c_s}$. Thus the outer part of the beam transfer function is indeed irrational, incorporating the essential singularities.

Following the technique in [Callier and Desoer, p. 655], we can readily compute a coprime factorization of this plant as

$$P(s) = \frac{P_i(s)P_o(s)}{\frac{s}{s+1}} \tag{2.19}$$

and

$$\frac{1}{P_i(0)P_o(0)} \cdot P_i(s)P_o(s) + \frac{s}{s+1} \cdot \frac{(s+1) - \frac{(s+1)P_i(s)P_o(s)}{P_i(0)P_o(0)}}{s} = 1 \tag{2.20}$$

2.1.5 Asymptotic Behavior of the Transfer Function

We now examine the behavior of the transfer function at infinity, *i.e.*, whether the transfer function is strictly proper. Our main point here is to show that it decreases at infinity in the closed right half plane faster than any polynomial.

The demonstration of this fact follows two observations:

1. The image of the right half plane under the map

$$s \mapsto a = \left(-\frac{ms^2 + cs}{EI + c_s I s}\right)^{\frac{1}{4}} \tag{2.21}$$

   satisfies $|\mathrm{Im}(a)| \leq \mathrm{Re}(a)$ and approaches the cone $\frac{\pi}{8} \leq |\arg(a)| \leq \frac{\pi}{4}$ in the right half plane uniformly in $\arg(s)$ for large $|s|$. As $|s| \to \infty$, $|a| \to \infty$.

2. Let $a = x + jy$ with $x, y \in \mathbf{R}$. Then

$$\left|EIa^2 \cdot P(s(a))\right|^2 = \frac{-\begin{pmatrix} 8\cos(x+y)\cosh(x+y) + 8\cos(x-y)\cosh(x-y) \\ -4\cosh(2y) - 4\cos(2y) - 4\cosh(2x) - 4\cos(2x) \end{pmatrix}}{\begin{pmatrix} \cosh(2x+2y) + \cos(2x+2y) + \cosh(2x-2y) + \cos(2x-2y) \\ -8\cos(x-y)\cosh(x+y) - 8\cos(x+y)\cosh(x-y) \\ +2\cos(2y)\cosh(2y) + 2\cos(2x)\cosh(2x) + 8 \end{pmatrix}} \tag{2.22}$$

From observation (1) any a in the image of the right half plane satisfies $y = r_a x$, with $r_a \leq 1$. Again by observation (1), for sufficiently large $|s|$, r_a is bounded from zero. Then using observation (2) it is easy to check that for a sufficiently large in the right half plane

$|P(s(a))|^2$ is bounded above by a positive constant times e^{-2ra} for some fixed $r > 0$. This implies that any polynomial in $|s|$ times $|P(s)|$ goes to zero at infinity in the right half plane. Without much more work one can in fact check that $|P(j\omega)|$ is asymptotically equal to

$$\frac{1}{E}\sqrt{\frac{c_s}{mI\,|\omega|}}\cdot e^{-k\sqrt[4]{|\omega|}} \tag{2.23}$$

where $k = \sqrt[4]{\frac{m}{c_s I}}\sin\frac{\pi}{8}$. This can be checked in the plots in Figures 2.3 and 2.4 (which use the same parameters for the beam model as above).

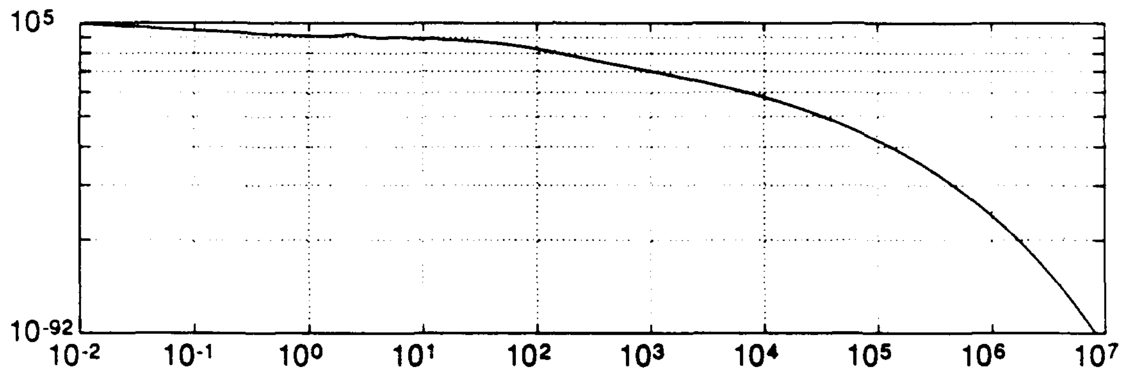

Figure 2.3: $|P(s)|$ along the imaginary axis.

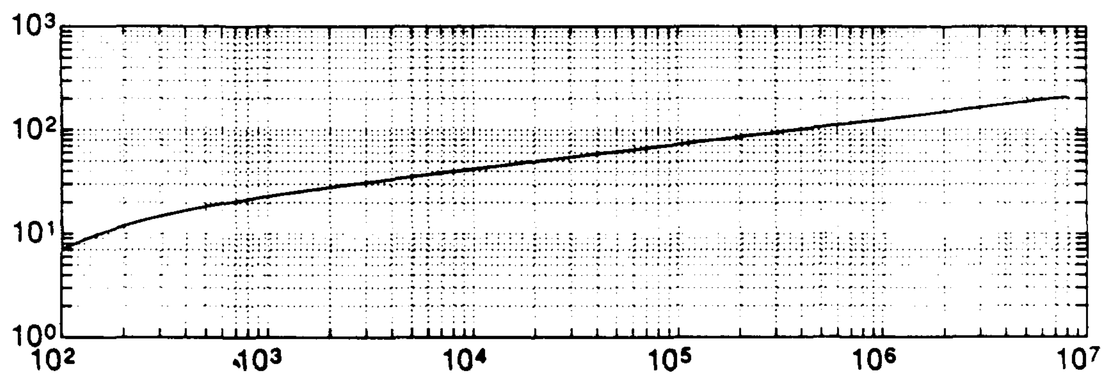

Figure 2.4: $\log|P(s)|$ along the imaginary axis.

2.2 A Damped Two-Degree-of-Freedom Beam

The model for a flexible beam with a tip body, including both bending and torsional vibrations consists of a pair of decoupled partial differential equations with coupled boundary equations. The partial differential equations are [41]:

$$\frac{\partial^2 y(x,t)}{\partial t^2} + 2\delta\frac{EI}{\rho}\frac{\partial^5 y(x,t)}{\partial t\partial x^4} + \frac{EI}{\rho}\frac{\partial^4 y(x,t)}{\partial x^4} = -x\theta''(t) \tag{2.24}$$

$$\frac{\partial^2\phi(x,t)}{\partial t^2} - 2\delta\frac{GJ}{\rho\kappa^2}\frac{\partial^3\phi(x,t)}{\partial t\partial x^2} - \frac{GJ}{\rho\kappa^2}\frac{\partial^2\phi(x,t)}{\partial x^2} = 0 \tag{2.25}$$

where y(x,t) represents the transverse displacement of the beam and $\phi(x,t)$ represents the angle of twist of the beam. The angle of rotation of the motor is given by $\theta(t)$; consequently $\theta''(t)$ represents the angular acceleration of the motor.

Taking Laplace transforms of each equation and using zero initial conditions (since the transfer function is derived from a zero resting state) yields:

$$s^2Y(x,s) + \frac{EI}{\rho}(2\delta s+1)Y^{(4)}(x,s) = -xs^2\Theta(s) \tag{2.26}$$

$$s^2\Phi(x,s) - \frac{GJ}{\rho\kappa^2}(2\delta s+1)\Phi^{(2)}(x,s) = 0 \tag{2.27}$$

Both equations are separable and therefore have solutions of the form:

$$Y(x,s) = w(s)v(x)$$

$$\Phi(x,s) = \beta(s)\alpha(x)$$

Using these relationships in equations 2.26 and 2.27 yields:

$$\lambda_1^4 = \frac{v^{(4)}(x)}{v(x)} = \frac{-s^2\rho}{EI(2\delta s+1)} \tag{2.28}$$

$$\lambda_2^2 = \frac{\alpha^{(2)}(x)}{\alpha(x)} = \frac{s^2\rho\kappa^2}{GJ(2\delta s+1)} \tag{2.29}$$

Note: the solution given is for the associated homogeneous equation to equation 2.26; equation 2.27 is already homogeneous

Solutions to the equations 2.26 and 2.27 are therefore of the form:

$$Y(x,s) = (k_1\cos(\lambda_1 x) + k_2\sin(\lambda_1 x) + k_3\cosh(\lambda_1 x) + k_4\sinh(\lambda_1 x))w(s) + y_p(x,s) \tag{2.30}$$

$$\Phi(x,s) = (a_1e^{\lambda_2 x} + a_2e^{-\lambda_2 x})\beta(s) \tag{2.31}$$

where $y_p(x,s)$ is a particular solution of the nonhomogeneous equation 2.26. A solution to equation 2.26 is given by $y_p(x,s) = -x\Theta(s)$ so we have:

$$Y(x,s) = (k_1\cos(\lambda_1 x) + k_2\sin(\lambda_1 x) + k_3\cosh(\lambda_1 x) + k_4\sinh(\lambda_1 x))w(s) - x\Theta(s) \tag{2.32}$$

The boundary conditions are related to the k_i and a_i by:

$$\begin{pmatrix} Y(0,s) \\ Y'(0,s) \\ Y''(0,s) \\ Y'''(0,s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_1 & 0 & \lambda_1 \\ -\lambda_1^2 & 0 & \lambda_1^2 & 0 \\ 0 & -\lambda_1^3 & 0 & \lambda_1^3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} w(s) - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Theta(s) \tag{2.33}$$

and

$$\begin{pmatrix} Y(L,s) \\ Y'(L,s) \\ Y''(L,s) \\ Y'''(L,s) \end{pmatrix} = \begin{pmatrix} \cos(\lambda_1 L) & \sin(\lambda_1 L) & \cosh(\lambda_1 L) & \sinh(\lambda_1 L) \\ -\lambda_1 \sin(\lambda_1 L) & \lambda_1 \cos(\lambda_1 L) & \lambda_1 \sinh(\lambda_1 L) & \lambda_1 \cosh(\lambda_1 L) \\ -\lambda_1^2 \cos(\lambda_1 L) & -\lambda_1^2 \sin(\lambda_1 L) & \lambda_1^2 \cosh(\lambda_1 L) & \lambda_1^2 \sinh(\lambda_1 L) \\ \lambda_1^3 \sin(\lambda_1 L) & -\lambda_1^3 \cos(\lambda_1 L) & \lambda_1^3 \sinh(\lambda_1 L) & \lambda_1^3 \cosh(\lambda_1 L) \end{pmatrix}$$

$$\cdot \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} w(s) - \begin{pmatrix} L \\ 1 \\ 0 \\ 0 \end{pmatrix} \Theta(s) \tag{2.34}$$

for the transverse displacement and

$$\begin{pmatrix} \Phi(0,s) \\ \Phi'(0,s) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_2 & -\lambda_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \beta(s) \tag{2.35}$$

and

$$\begin{pmatrix} \Phi(L,s) \\ \Phi'(L,s) \end{pmatrix} = \begin{pmatrix} e^{\lambda_2 L} & e^{-\lambda_2 L} \\ \lambda_2 e^{\lambda_2 L} & -\lambda_2 e^{-\lambda_2 L} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \beta(s) \tag{2.36}$$

for the angle of twist.

Next we combine equations 2.33 and 2.34 to eliminate the k_i and $w(s)$, resulting in:

$$\begin{pmatrix} Y(L,s) \\ Y'(L,s) \\ Y''(L,s) \\ Y'''(L,s) \end{pmatrix} = \begin{pmatrix} \frac{\sinh(\lambda_1 L)+\sin(\lambda_1 L)}{2\lambda_1} - L \\ \frac{\cosh(\lambda_1 L)+\cos(\lambda_1 L)}{2} - 1 \\ \frac{\lambda_1 \sinh(\lambda_1 L)-\sin(\lambda_1 L)}{2} \\ \frac{\lambda_1^2 \cosh(\lambda_1 L)-\cos(\lambda_1 L)}{2} \end{pmatrix} + \frac{1}{2} \cdot$$

$$\begin{pmatrix} \cos(\lambda_1 L)+\cosh(\lambda_1 L) & \frac{\sinh(\lambda_1 L)+\sin(\lambda_1 L)}{\lambda_1} & \frac{\cosh(\lambda_1 L)-\cos(\lambda_1 L)}{\lambda_1^2} & \frac{\sinh(\lambda_1 L)-\sin(\lambda_1 L)}{\lambda_1^3} \\ \lambda_1(\sinh(\lambda_1 L)-\sin(\lambda_1 L)) & \cos(\lambda_1 L)+\cosh(\lambda_1 L) & \frac{\sinh(\lambda_1 L)+\sin(\lambda_1 L)}{\lambda_1} & \frac{\cosh(\lambda_1 L)-\cos(\lambda_1 L)}{\lambda_1^2} \\ \lambda_1^2(\cosh(\lambda_1 L)-\cos(\lambda_1 L)) & \lambda_1(\sinh(\lambda_1 L)-\sin(\lambda_1 L)) & \cos(\lambda_1 L)+\cosh(\lambda_1 L) & \frac{\sinh(\lambda_1 L)+\sin(\lambda_1 L)}{\lambda_1} \\ \lambda_1^3(\sinh(\lambda_1 L)+\sin(\lambda_1 L)) & \lambda_1^2(\cosh(\lambda_1 L)-\cos(\lambda_1 L)) & \lambda_1(\sinh(\lambda_1 L)-\sin(\lambda_1 L)) & \cos(\lambda_1 L)+\cosh(\lambda_1 L) \end{pmatrix}$$

$$\cdot \begin{pmatrix} Y(0,s) \\ Y'(0,s) \\ Y''(0,s) \\ Y'''(0,s) \end{pmatrix} \tag{2.37}$$

Similarly, combining equations 2.35 and 2.36 to eliminate the a_i and $\beta(s)$ we obtain:

$$\begin{pmatrix} \Phi(L,s) \\ \Phi'(L,s) \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} e^{\lambda_2 L} + e^{-\lambda_2 L} & \frac{1}{\lambda_2}(e^{\lambda_2 L} - e^{-\lambda_2 L}) \\ \lambda_2(e^{\lambda_2 L} - e^{-\lambda_2 L}) & e^{\lambda_2 L} + e^{-\lambda_2 L} \end{pmatrix} \begin{pmatrix} \Phi(0,s) \\ \Phi'(0,s) \end{pmatrix} \tag{2.38}$$

The simple boundary conditions are given by:

$$\left.\begin{aligned} Y(0,s) &= 0 \\ Y'(0,s) &= 0 \\ \Phi(0,s) &= 0 \end{aligned}\right\} \tag{2.39}$$

while the coupled boundary conditions are [41]:

$$
\begin{aligned}
0 &= ms^2[(L+c)\Theta(s) + Y(L,s) + cY'(L,s) + e\Phi(L,s)] - EI(1+2\delta s)Y'''(L,s) \quad (2.40)\\
0 &= mcs^2[(L+c)\Theta(s) + Y(L,s) + cY'(L,s) + e\Phi(L,s)]\\
&\quad + J_o s^2[\Theta(s) + Y'(L,s)] + EI(1+2\delta s)Y''(L,s) \quad (2.41)\\
0 &= mes^2[(L+c)\Theta(s) + Y(L,s) + cY'(L,s) + e\Phi(L,s)]\\
&\quad + J_e s^2\Phi(L,s) + GJ(1+2\delta s)\Phi'(L,s) \quad (2.42)
\end{aligned}
$$

Now, using equations 2.37 and 2.38 with the boundary conditions given above, we have 9 equations in 10 unknowns. All that remains to be done to find the transfer function is to eliminate 8 of the variables. There are in fact two transfer functions of interest in this system. One is the transfer function from $\Theta(s)$, the angle of rotation of the motor, to $Y(L,s)$, the transverse displacement at the end of the beam; the other is the transfer function from $\Theta(s)$ to $\Phi(L,s)$, the angle of twist at the end of the beam.

Using Macsyma, these transfer functions were found to be:

$$\frac{Y(L,s)}{\Theta(s)} = \frac{N_1(s)}{D(s)} \quad (2.43)$$

$$\frac{\Phi(L,s)}{\Theta(s)} = \frac{N_2(s)}{D(s)} \quad (2.44)$$

where $N_1(s)$, $N_2(s)$ and $D(s)$ are given by:

$$
\begin{aligned}
N_1(s) = &-[s^2 J_e \sinh(\lambda_2 L) + \lambda_2 GJ(2\delta s+1)\cosh(\lambda_2 L)]\{\lambda_1^3 E^2 I^2 (2\delta s+1)^2[\sinh(\lambda_1 L) + \sin(\lambda_1 L) -\\
&\lambda_1 L(\cos(\lambda_1 L)\cosh(\lambda_1 L) + 1)] - 2\lambda_1^2 EIL(2\delta s+1)mcs^2(\sin(\lambda_1 L)\sinh(\lambda_1 L)) +\\
&\lambda_1 EI(2\delta s+1)mcs^2(\sinh(\lambda_1 L) - \sin(\lambda_1 L)) + \lambda_1 EIL(2\delta s+1)ms^2(\cos(\lambda_1 L)\sinh(\lambda_1 L) -\\
&\cosh(\lambda_1 L)\sin(\lambda_1 L)) + \lambda_1^2 EI(2\delta s+1)s^2(mc^2 + J_0)(\cosh(\lambda_1 L) - \cos(\lambda_1 L)) -\\
&\lambda_1^3 EIL(2\delta s+1)s^2(mc^2+J_0)(\cos(\lambda_1 L)\sinh(\lambda_1 L) + \cosh(\lambda_1 L)\sin(\lambda_1 L)) +\\
&J_0 L m s^4(\cos(\lambda_1 L)\cosh(\lambda_1 L) - 1)\} - \lambda_1^2 EI\sinh(\lambda_2 L)s^2(2\delta s+1)me^2\{s^2 J_0[\cosh(\lambda_1 L) -\\
&\cos(\lambda_1 L) - \lambda_1 L(\cos(\lambda_1 L)\sinh(\lambda_1 L) + \cosh(\lambda_1 L)\sin(\lambda_1 L))] + \lambda_1 EI(2\delta s+1)[\sinh(\lambda_1 L) +\\
&\sin(\lambda_1 L) - \lambda_1 L(\cos(\lambda_1 L)\cosh(\lambda_1 L) + 1)]\}
\end{aligned}
\quad (2.45)
$$

$$
\begin{aligned}
N_2(s) = &\lambda_1^2 EI\sinh(\lambda_2 L)mes^2(2\delta s+1)\{J_0 s^2(\cosh(\lambda_1 L) - \cos(\lambda_1 L)) + (2\delta s+1)\lambda_1 EI\cdot\\
&[\sinh(\lambda_1 L) + \sin(\lambda_1 L) + \lambda_1 c(\cosh(\lambda_1 L) + \cos(\lambda_1 L))]\}
\end{aligned}
\quad (2.46)
$$

$$
\begin{aligned}
D(s) = &[s^2 J_e \sinh(\lambda_2 L) + \lambda_2 GJ(2\delta s+1)\cosh(\lambda_2 L)]\{ms^4 J_0(\cos(\lambda_1 L)\cosh(\lambda_1 L) - 1) -\\
&(2\delta s+1)^2\lambda_1^4 E^2 I^2(\cos(\lambda_1 L)\cosh(\lambda_1 L) + 1) + ms^2\lambda_1 EI(2\delta s+1)[\cos(\lambda_1 L)\sinh(\lambda_1 L) -\\
&\cosh(\lambda_1 L)\sin(\lambda_1 L) - 2\lambda_1 c\sin(\lambda_1 L)\sinh(\lambda_1 L)] - (mc^2+J_0)\lambda_1^3 EIs^2(2\delta s+1)\cdot\\
&(\cos(\lambda_1 L)\sinh(\lambda_1 L) + \cosh(\lambda_1 L)\sin(\lambda_1 L))\} - me^2 s^2\sinh(\lambda_2 L)\lambda_1^3 EI(2\delta s+1)\cdot\\
&[\lambda_1 EI(2\delta s+1)(1 + \cos(\lambda_1 L)\cosh(\lambda_1 L)) + J_0 s^2(\cos(\lambda_1 L)\sinh(\lambda_1 L) + \cosh(\lambda_1 L)\sin(\lambda_1 L))]
\end{aligned}
\quad (2.47)
$$

2.3 A Damped Timoshenko Beam

The partial differential equation model, including damping terms, for the Timoshenko beam is given by:

$$\frac{\partial^4 v}{\partial x^4} + \frac{c_s}{E}\frac{\partial^5 v}{\partial x^4 \partial t} - \left(\frac{\rho}{KG} + \frac{\rho}{E}\right)\frac{\partial^4 v}{\partial x^2 \partial t^2} + c\frac{\partial v}{\partial t} + \frac{\rho}{EK_r^2}\frac{\partial^2 v}{\partial t^2} + \frac{\rho^2}{KEG}\frac{\partial^4 v}{\partial t^4} = 0 \qquad (2.48)$$

let

$$\frac{c_s}{E} = c_1, \quad -\left(\frac{\rho}{KG} + \frac{\rho}{E}\right) = c_2, \quad c = c_3, \quad \frac{\rho}{EK_r^2} = c_4, \quad \frac{\rho^2}{KEG} = c_5$$

By taking the Laplace transform of the equation we obtain:

$$(1 + c_1 s)v^{(4)} + c_2 s^2 v^{(2)} + (c_5 s^4 + c_4 s^2 + c_3 s)v = 0 \qquad (2.49)$$

where $v = v(x, s)$ and primes denote differentiation with respect to x. This is an ordinary differential equation. Making the usual assumption that the solutions have the form $c(s)e^{\lambda(s)x}$ and applying the differential operator we have:

$$(1 + c_1 s)\lambda^4 + c_2 s^2 \lambda^2 + (c_5 s^4 + c_4 s^2 + c_3 s) = 0$$

where the roots of this fourth order equation yield four possible values of $\lambda(s)$. Solving this, we discover that the solutions take the form:

$$v(x, s) = k_1(s)\cosh\lambda_1 x + k_2(s)\sinh\lambda_1 x + k_3(s)\cosh\lambda_2 x + k_4(s)\sinh\lambda_2 x \qquad (2.50)$$

where

$$\lambda_1 = \left[\frac{-c_2 s^2 + (-4c_1c_5 s^5 + (c_2^2 - 4c_5)s^4 - 4(c_1c_4)s^3 - 4(c_4 + c_1c_3)s^2 - 4c_3 s)^{\frac{1}{2}}}{2(1 + c_1 s)}\right]^{\frac{1}{2}} \qquad (2.51)$$

$$\lambda_2 = \left[\frac{-c_2 s^2 - (-4c_1c_5 s^5 + (c_2^2 - 4c_5)s^4 - 4(c_1c_4)s^3 - 4(c_4 + c_1c_3)s^2 - 4c_3 s)^{\frac{1}{2}}}{2(1 + c_1 s)}\right]^{\frac{1}{2}}$$

The boundary conditions at each end of the beam ($x = 0$, $x = L$) are related to $k_1(s) \ldots k_4(s)$ by:

$$\begin{bmatrix} v(0,s) \\ v'(0,s) \\ v''(0,s) \\ v'''(0,s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_1 & 0 & \lambda_2 \\ \lambda_1^2 & 0 & \lambda_2^2 & 0 \\ 0 & \lambda_1^3 & 0 & \lambda_2^3 \end{bmatrix} \begin{bmatrix} k_1(s) \\ k_s(s) \\ k_3(s) \\ k_4(s) \end{bmatrix}$$

and

$$\begin{bmatrix} v(L,s) \\ v'(L,s) \\ v''(L,s) \\ v'''(L,s) \end{bmatrix} = \begin{bmatrix} \cosh\lambda_1 L & \sinh\lambda_1 L & \cosh\lambda_2 L & \sinh\lambda_2 L \\ \lambda_1\sinh\lambda_1 L & \lambda_1\cosh\lambda_1 L & \lambda_2\sinh\lambda_2 L & \lambda_2\cosh\lambda_2 L \\ \lambda_1^2\cosh\lambda_1 L & \lambda_1^2\sinh\lambda_1 L & \lambda_2^2\cosh\lambda_2 L & \lambda_2^2\sinh\lambda_2 L \\ \lambda_1^3\sinh\lambda_1 L & \lambda_1^3\cosh\lambda_1 L & \lambda_2^3\sinh\lambda_2 L & \lambda_3^3\cosh\lambda_2 L \end{bmatrix} \begin{bmatrix} k_1(s) \\ k_2(s) \\ k_3(s) \\ k_4(s) \end{bmatrix}$$

The relationship between boundary conditions at either end is:

$$\begin{bmatrix} v(L,s) \\ v'(L,s) \\ v''(L,s) \\ v'''(L,s) \end{bmatrix} = T \begin{bmatrix} v(0,s) \\ v'(0,s) \\ v''(0,s) \\ v'''(0,s) \end{bmatrix}$$

$$T = \left(\frac{1}{\lambda_2^2 - \lambda_1^2}\right)$$

$$\cdot \left[\begin{matrix} \lambda_2^2 \cosh \lambda_1 L - \lambda_1^2 \cosh \lambda_2 L & \frac{\lambda_2^2}{\lambda_1} \sinh \lambda_1 L - \frac{\lambda_1^2}{\lambda_2} \sinh \lambda_2 L \\ \lambda_1 \lambda_2^2 \sinh \lambda_1 L - \lambda_1^2 \lambda_2 \sinh \lambda_2 L & \lambda_2^2 \cosh \lambda_1 L - \lambda_1^2 \cosh \lambda_2 L \\ \lambda_1^2 \lambda_2^2 \cosh \lambda_1 L - \lambda_1^2 \lambda_2^2 \cosh \lambda_2 L & \lambda_1 \lambda_2^2 \sinh \lambda_1 L - \lambda_1^2 \lambda_2 \sinh \lambda_2 L \\ \lambda_1^3 \lambda_2^2 \sinh \lambda_1 L - \lambda_1^2 \lambda_2^3 \sinh \lambda_2 L & \lambda_1^2 \lambda_2^2 \cosh \lambda_1 L - \lambda_1^2 \lambda_2^2 \cosh \lambda_2 L \end{matrix}\right.$$

$$\left.\begin{matrix} \cosh \lambda_2 L - \cosh \lambda_1 L & \frac{1}{\lambda_2} \sinh \lambda_2 L - \frac{1}{\lambda_1} \sinh \lambda_1 L \\ \lambda_2 \sinh \lambda_2 L - \lambda_1 \sinh \lambda_1 L & \cosh \lambda_2 L - \cosh \lambda_1 L \\ \lambda_2^2 \cosh \lambda_2 L - \lambda_1^2 \cosh \lambda_1 L & \lambda_2 \sinh \lambda_2 L - \lambda_1 \sinh \lambda_1 L \\ \lambda_2^3 \sinh \lambda_2 L - \lambda_1^3 \sinh \lambda_1 L & \lambda_2^2 \cosh \lambda_2 L - \lambda_1^2 \cosh \lambda_1 L \end{matrix}\right]$$

The boundary conditions considered are:

$$\left\{ \begin{array}{rcl} v''(0,s) - \frac{\rho}{KG} s^2 v(0,s) & = & \frac{T(s)}{EI} \\ \frac{E}{\rho} v'''(0,s) - \left[1 + \frac{E}{KG}\right] s^2 v'(0,s) & = & 0 \\ v''(L,s) - \frac{\rho}{KG} s^2 v(L,s) & = & \frac{T(s)}{EI} \\ \frac{E}{\rho} v'''(L,s) - \left[1 + \frac{E}{KG}\right] s^2 v'(L,s) & = & 0 \end{array} \right.$$

The relationship between the boundary conditions given by the matrix T along with the boundary conditions themselves define a system of eight equations in nine unknowns. If we solve these equations to get $v(0,s)$ in terms of the torque $T(s)$, then we have the transfer function from the torque applied at one end to the displacement at the other end. The solution was performed on Macsyma, and yielded the transfer function:

$$P(s) = \frac{v(0,s)}{T(s)} = \frac{N(s)}{D(s)}$$

where:

$$\begin{aligned} N(s) \;=\; & -GK((K + EG)(GK + E)((\lambda_2^2 + \lambda_1^2) \sinh(\lambda_1 L) \sinh(\lambda_2 L) + \\ & 2\lambda_1 \lambda_2 (1 - \cosh(\lambda_1 L) \cosh(\lambda_2 L)))\rho^3 s^6 + K(-(\lambda_2^3 + \lambda_1^3) EG(K + EG) \cdot \\ & (\lambda_2 \sinh^2(\lambda_2 L) + \lambda_1 \sinh^2(\lambda_1 L)) - (2\lambda_1^2 \lambda_2^2 G^2 K(K + EG) + (\lambda_2^2 + \lambda_1^2) \cdot \\ & (\lambda_2^2 - \lambda_1 \lambda_2 + \lambda_1^2) EGK - \lambda_1 \lambda_2 (\lambda_2 - \lambda_1)^2 E^2 G^2 + (\lambda_2^4 + \lambda_1^4) E^2) \cdot \\ & \sinh(\lambda_1 L) \sinh(\lambda_2 L) + \lambda_1 \lambda_2 (\lambda_2^2 + \lambda_1^2)(G(K + EG)(GK + 2E) + \end{aligned}$$

$$E(GK+E))(\cosh(\lambda_1 L)\cosh(\lambda_2 L)-1)+\lambda_1\lambda_2(\lambda_2^2-\lambda_1^2)G(K+EG)\cdot$$
$$(GK+E)(\cosh(\lambda_2 L)-\cosh(\lambda_1 L)))\rho^2 s^4+EGK^2((\lambda_2^3+\lambda_1^3)E\cdot$$
$$(\lambda_2^3\sinh^2(\lambda_2 L)+\lambda_1^3\sinh^2(\lambda_1 L))+\lambda_1^2\lambda_2^2((\lambda_2^2+\lambda_1^2)G(2K+EG)+$$
$$(\lambda_2-\lambda_1)^2E)\sinh(\lambda_1 L)\sinh(\lambda_2 L)+\lambda_1\lambda_2(-E(\lambda_2^4\cosh(\lambda_2 L)+\lambda_1^4\cosh(\lambda_1 L))+$$
$$EG^2(\lambda_1^4\cosh(\lambda_2 L)+\lambda_2^4\cosh(\lambda_1 L))+\lambda_1^2\lambda_2^2E(1-G^2)(\cosh(\lambda_2 L)+\cosh(\lambda_1 L))+$$
$$(\lambda_2^4-\lambda_1^4)GK(\cosh(\lambda_1 L)-\cosh(\lambda_2 L)))-\lambda_1\lambda_2((\lambda_2^2+\lambda_1^2)^2GK+$$
$$(\lambda_2^4+\lambda_1^4)EG^2+4\lambda_1^2\lambda_2^2E)\cosh(\lambda_1 L)\cosh(\lambda_2 L)+\lambda_1\lambda_2((\lambda_2^2+\lambda_1^2)^2(GK+E)+$$
$$2\lambda_1^2\lambda_2^2EG^2))\rho s^2+\lambda_1^2\lambda_2^2E^2G^2K^3(-(\lambda_2^4+\lambda_1^4)\sinh(\lambda_1 L)\sinh(\lambda_2 L)+\lambda_1\lambda_2\cdot$$
$$(\lambda_2^2+\lambda_1^2)(\cosh(\lambda_1 L)\cosh(\lambda_2 L)-1)+\lambda_1\lambda_2(\lambda_2^2-\lambda_1^2)(\cosh(\lambda_2 L)-\cosh(\lambda_1 L))))$$

$$
\begin{aligned}
D(s) = {} & EI((K+EG)(GK+E)((\lambda_2^2+\lambda_1^2)\sinh(\lambda_1 L)\sinh(\lambda_2 L)+ \\
& 2\lambda_1\lambda_2(1-\cosh(\lambda_1 L)\cosh(\lambda_2 L)))\rho^4 s^8+K(-(\lambda_2^3+\lambda_1^3)EG(K+EG)\cdot \\
& (\lambda_2\sinh^2(\lambda_2 L)+\lambda_1\sinh^2(\lambda_1 L))+(-(\lambda_2^4+\lambda_1^4)E(GK+E)- \\
& 4\lambda_1^2\lambda_2^2G^2K(K+EG)+\lambda_1\lambda_2(\lambda_2^2-4\lambda_1\lambda_2+\lambda_1^2)EG(K+EG)) \\
& \sinh(\lambda_1 L)\sinh(\lambda_2 L)+\lambda_1\lambda_2(\lambda_2^2+\lambda_1^2)(2G^2K(K+EG)+ \\
& 4EGK+3E^2G^2+E^2)(\cosh(\lambda_1 L)\cosh(\lambda_2 L)-1))\rho^3 s^6+GK^2((\lambda_2^3+\lambda_1^3)\cdot \\
& E(E(\lambda_2^3\sinh^2(\lambda_2 L)+\lambda_1^3\sinh^2(\lambda_1 L))+\lambda_1\lambda_2G(K+EG)\cdot \\
& (\lambda_1\sinh^2(\lambda_2 L)+\lambda_2\sinh^2(\lambda_1 L)))+\lambda_1\lambda_2(\lambda_1\lambda_2(\lambda_2^2+\lambda_1^2)G^2K(K+EG)- \\
& (\lambda_2^4-4\lambda_1\lambda_2(\lambda_2^2+\lambda_1^2)+\lambda_1^4)EGK-(\lambda_2^4-2\lambda_1\lambda_2^3-2\lambda_1^3\lambda_2+\lambda_1^4)E^2G^2 \\
& +2\lambda_1\lambda_2(\lambda_2^2-\lambda_1\lambda_2+\lambda_1^2)E^2)\sinh(\lambda_1 L)\sinh(\lambda_2 L)+\lambda_1\lambda_2(2\lambda_1^2\lambda_2^2G^2K(K+EG)+ \\
& 2(\lambda_2^4+3\lambda_1^2\lambda_2^2+\lambda_1^4)EGK+(\lambda_2^4+4\lambda_1^2\lambda_2^2+\lambda_1^4)E^2(G^2+1))\cdot \\
& (1-\cosh(\lambda_1 L)\cosh(\lambda_2 L)))\rho^2 s^4+\lambda_1^2\lambda_2^2EG^2K^3(-(\lambda_2^3+\lambda_1^3)E\cdot \\
& (\lambda_2\sinh^2(\lambda_2 L)+\lambda_1\sinh^2(\lambda_1 L))-(2\lambda_1^2\lambda_2^2G(2K+EG)+ \\
& (\lambda_2^4-\lambda_1\lambda_2^3+2\lambda_1^2\lambda_2^2-\lambda_1^3\lambda_2+\lambda_1^4)E)\sinh(\lambda_1 L)\sinh(\lambda_2 L)+\lambda_1\lambda_2(\lambda_2^2+\lambda_1^2)\cdot \\
& (G(2K+EG)+3E)(\cosh(\lambda_1 L)\cosh(\lambda_2 L)-1))\rho s^2+\lambda_1^4\lambda_2^4E^2G^3K^4\cdot \\
& ((\lambda_2^2+\lambda_1^2)\sinh(\lambda_1 L)\sinh(\lambda_2 L)+2\lambda_1\lambda_2(1-\cosh(\lambda_1 L)\cosh(\lambda_2 L))))
\end{aligned}
$$

If we use the relationship between λ_1 and λ_2, some further simplification is possible. For example, if we write $\lambda_1 = (u(s)+w(s))^{\frac{1}{2}}$, we have $\lambda_2 = (u(s)-w(s))^{\frac{1}{2}}$. Then we observe:

$$
\begin{aligned}
\lambda_1^2+\lambda_2^2 &= 2u(s) \\
\lambda_1^2-\lambda_2^2 &= 2w(s) \\
\lambda_1^2\lambda_2^2 &= u^2(s)-w^2(s) \\
\lambda_1^4+\lambda_2^4 &= u^2(s)+w^2(s) \\
\lambda_1^2\lambda_2^4+\lambda_1^4\lambda_2^2 &= \lambda_1^2\lambda_2^2(\lambda_1^2+\lambda_2^2) = 2u^3(s)-2uw^2(s)
\end{aligned}
$$

These substitutions can be made to simplify $N(s)$ and $D(s)$ as functions of s. However, such simplification would introduce complicated functions of s into the equation so we prefer to leave the representation in terms of λ_1 and λ_2.

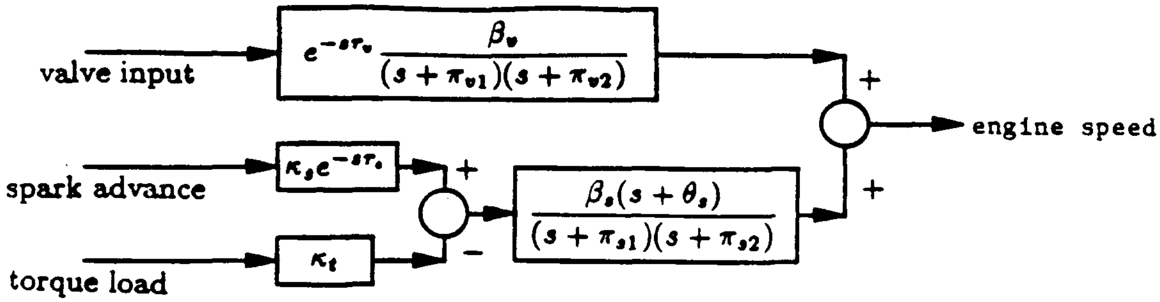

Figure 2.5: Engine Model Structure

2.4 A Multivariable Delay Problem

In [46], the idle speed control of a typical V6 fuel-injected engine with computer controlled management system was considered. The model structure consists of delays and second order dynamics. The valve-to-rpm transfer function contains no zeros. The spark-to-rpm transfer function has one zero. The system is described by Figure 2.5.

From Figure 2.5 we see

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p_1 & p_2 & p_3 \\ p_1 & p_2 & p_3 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

where

$$p_1 = -\frac{\beta_s(s+\theta_s)}{(s+\pi_{s1})(s+\pi_{s2})}\kappa_t, p_2 = \frac{\beta_v}{(s+\pi_{v1})(s+\pi_{v2})}e^{-s\tau_v}, p_3 = \frac{\beta_s(s+\theta_s)}{(s+\pi_{s1})(s+\pi_{s2})}\kappa_s e^{-s\tau_s}.$$

We primarily will be interested in the transfer function from d to e, which we denote by

$$f_{ed} = \begin{pmatrix} f_{e_1 d} \\ f_{e_2 d} \\ f_{e_3 d} \end{pmatrix} = P_{11} + P_{12}K(1 - P_{22}K)^{-1}P_{21} = \begin{pmatrix} 0 \\ 0 \\ p_1 \end{pmatrix} + \frac{p_1}{1 - p_2 k_1 - p_3 k_2} \begin{pmatrix} k_1 \\ k_2 \\ p_2 k_1 + p_3 k_2 \end{pmatrix}.$$

Chapter 3

Outer Factor Absorption

3.1 Introduction

A crucial step in the derivation of the solution to $\mathbf{H}^\infty$ optimal control problems is to show that the outer factor of the plant can be ignored at a certain point in the calculation of the infimal norm of the $\mathbf{H}^\infty$ control criterion. Specifically, given the expressions

$$\mu_o = \inf_{Q \in \mathbf{H}^\infty} \|W - FMQ\|_\infty$$

and

$$\mu_i = \inf_{Q \in \mathbf{H}^\infty} \|W - MQ\|_\infty$$

with $W \in \mathbf{L}^\infty$, $M \in \mathbf{H}^\infty$ inner and $F \in \mathbf{H}^\infty$ outer, the assumpton is made that $\mu_o = \mu_i$, and computation proceeds to find μ_i.

The validity of this step has been demonstrated for rational plants [49] and for plants with irrational inner part but rational outer and unstable parts [10],[18].

Our recent study of an example [11] has lead us to be concerned with plants having irrational outer part. In the present work we extend the results in [10],[18] to the case of irrational outer factors for the plant.

The organization of this chapter is as follows:

In Section 3.2 we review how the $\mathbf{H}^\infty$ optimal weighted sensitivity problem for unstable plants can be reduced to a minimization problem which is affine in a free (functional) parameter, via the so-called "Q-parametrization." When the outer and unstable factors of the plant and the weighting function are rational, is known that under additional necessary assumptions on the weighting function a sequence of approximating solutions can be constructed by first solving a simpler problem in which an outer outer function has been "absorbed" into the free parameter, and then approximately "extracting" the outer factor. Proposition 1 summarizes these previously known necessary and sufficient conditions. For the beam model of Section 2.1 the outer factor of the plant is irrational. In this case the standard construction is [18] does not apply. In order to be able to treat still more general plants, we next define a general zero of a function, and proceed to prove corresponding necessary conditions in Lemma 1 of Section 3.3.

In Section 3.3 we present the main result of this chapter, Proposition 2, showing that the conditions of Lemma 1 are also sufficient under reasonable assumptions on the plant. The key ideas behind this extension of previous results are a construction to approximately invert irrational outer functions, and a definition of a generalized zero of a function which allows us to formulate satisfactory necessary and sufficient conditions for outer factor absorption.

3.2 Problem Definition and Background

Our main result here forms one step in the solution of $\mathbf{H}^\infty$ problems. For clarity's sake we first define the overall $\mathbf{H}^\infty$ problem which forms the setting of our result. This is the basic problem set forth in [52].

BASIC PROBLEM: Given a proper transfer function $P(s)$ (the plant), and a weighting function $W_1(s) \in \mathbf{H}^\infty$, find the infimal norm over all weighted stable disturbance sensitivity transfer functions attainable by stabilizing proper feedback, *i.e.*,

$$\inf_{C} \|W_1(s) \cdot S(s)\|_\infty,$$

where $C =$ {stabilizing and proper feedback compensators}.

In Figure 3.1, feedback connection for the problem is illustrated. $S(s) = (1 + PC)^{-1}$ is the transfer function from a disturbance $d(s)$ at the output of the plant to the closed loop output $y(s)$. The transfer function of the feedback compensator, $C(s)$, is constrained to be proper and stabilizing for the closed loop.

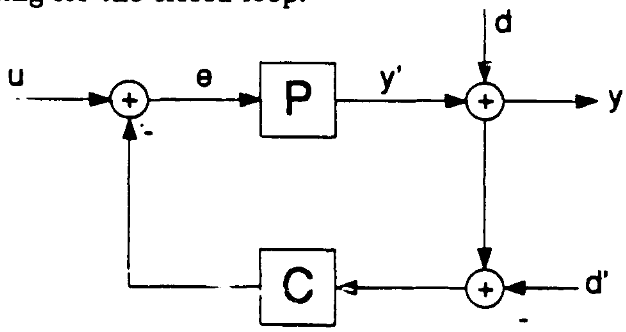

Figure 3.1: General Feedback System

Remark 1 *Since $\|M \cdot X\|_\infty = \|X\|_\infty$ for any inner function M and $X \in \mathbf{H}^\infty$, we assume (as usual) without loss of generality that W_1 is outer.*

As is well known, this problem has been solved under various additional assumptions. Here we separate the additional assumptions into two versions of the basic problem:

PROBLEM R (rational): Assume, in addition to the assumptions of Basic Problem, that $P(s)$ and $W_1(s)$ are rational functions.

PROBLEM I (general inner): Assume, in addition to the assumptions of Basic Problem, that $W_1(s)$ is a rational function, and $P(s) = \psi(s)\varphi(s)P_r(s)$, where

- $\psi(s)$ is a general inner function,
- $P_r(s)$ is rational
- $\varphi(s)$ is a general outer function in $\mathbf{H}^\infty$ with inverse in $\mathbf{H}^\infty$.

Remark 2 *In each of these problems, as a consequence of our assumptions one can factor the plant as $P(s) = P_i(s) \cdot P_o(s) \cdot P_u(s)$, an inner-outer-unstable factorization of the plant, where the unstable factor P_u is rational, analytic in the open left half plane, and bounded from zero in the open right half plane. We can assume without loss of generality that $P_u = D^{-1}$, with rational $D \in \mathbf{H}^\infty$. Analyticity of D^{-1} in the open left half plane implies that D has no zeros there, i.e., $D = D_i D_o$, where D_i is inner and D_o is outer with zeros (as a function on* C*) only on the (extended) imaginary axis.*

The (standard) first step in the solution involves the parametrization of all attainable disturbance sensitivity functions, which is as follows:

Theorem 1 *Given a plant P as above (Problems R or I) with factorization as in Remark 2, there exists an $\mathbf{H}^\infty$ function $V(s)$ such that, if the compensator C is proper and stabilizes the closed loop, then $Q \triangleq \frac{C}{1+PC}$ satisfies*

$$Q \in \mathbf{H}^\infty$$

and

$$(1 - PC)^{-1} = D(V - P_i P_o Q),$$

with P_i, P_o and D as in Remark 2 above. Conversely, if $Q \in \mathbf{H}^\infty$ then $(1-PC)^{-1} = D(V - P_i P_o Q) \in \mathbf{H}^\infty$, and $C = \frac{Q}{1+PQ}$ is a proper stabilizing compensator. Defining $\mathcal{C} \triangleq$ {proper stabilizing feedback compensators}, *we use this parametrization to conclude*

$$\inf_{C \in \mathcal{C}} \|W_1(1 + PC)^{-1}\|_\infty = \inf_{Q \in \mathbf{H}^\infty} \|W_1 DV - W_1 D P_i P_o Q\|_\infty \tag{3.1}$$

$$= \inf_{Q \in \mathbf{H}^\infty} \|W_1 D_o (V - P_i P_o Q)\|_\infty \tag{3.2}$$

Proof: See, for example, [23, p. 10]. V is the coefficient function in the Bezout identity giving a coprime factorization of $P : P_i P_o U + DV = 1$, and $P = P_i P_o / D$. ∎

As is well known, this theorem allows us to simplify notation, since (3.1) shows that finding a solution is equivalent to finding a solution to a related affine minimization problem. With this simplification, taking

$$\left.\begin{array}{rcl} W &=& W_1 DV \\ M &=& D_i P_i \\ F &=& W_1 D_0 P_0 \end{array}\right\} \tag{3.3}$$

the problems become equivalent to

$$\inf_{Q \in \mathbf{H}^\infty} \|W - FMQ\|_\infty \tag{3.4}$$

where,

- for Problem R, W, F and M are rational;
- for Problem I, W and M are not necessarily rational, and F is the product of a rational outer function with a stably invertible irrational outer function. In this case we can assume without loss of generality that F is rational, since the invertibility of the irrational outer factor of F allows us to ignore it in the minimization.

Previously known results can be summarized as follows

Proposition 1 *Let W_1 be rational and outer, and let $P = \psi\varphi P_r$ with ψ inner, φ an outer function in $\mathbf{H}^\infty$ invertible in $\mathbf{H}^\infty$, and P_r rational. Define*

$$\begin{aligned}\mu_o &= \inf_C \|W_1(s)\cdot(1+P(s)C(s))^{-1}\|_\infty \\ &= \inf_C \|W_1(s)D_o(s)(V(s)-P_i(s)P_o(s)Q(s))\|_\infty \\ &= \inf_C \|W(s)-F(s)M(s)Q(s)\|_\infty\end{aligned}$$

and

$$\begin{aligned}\mu_i &= \inf_C \|W(s)-M(s)Q(s)\|_\infty \\ &= \inf_{Q\in\mathbf{H}^\infty} \|W_1(s)D_o(s)V(s)-P_i(s)Q(s)\|_\infty.\end{aligned}$$

Let $\mathcal{V}(F)=\{z\in j\mathbf{R} : F(z)=0\}$. Then

$$\mu_i \geq W(z) \text{ for all } \quad z\in\mathcal{V}(F) \tag{3.5}$$

if and only if

$$\mu_o = \mu_i. \tag{3.6}$$

Proof: This is a special case of Proposition 2 below. ∎

Remark 3 *(3.5) is obviously a necessary condition for (3.6), so the essence of Proposition 1 is the sufficiency of (3.5) for (3.6). However, for more general F and W, the necessity of (3.5) is not certain: If P_o is discontinuous we first need to generalize the notion of zero. Then one might consider the case of W and F both beign discontinuous at a generalized zero of F. Other possible pathological cases are mentioned in the conclusion. In Lemma 1 below we proved the necessity for the relatively straightforward case of P_o continuous at its zeros.*

In considering $\mathbf{H}^\infty$ problems for distributed parameter systems, it has become clear that the assumptions of Problem I are too restrictive.

EXAMPLE: For the damped Euler-Bernoulli beam model of Section 2.1 the outer part has an essential singularity at infinity, and the transfer function rolls off faster than any rational function. (This is not just a result of poles clustering at ∞, since, for example, an infinite Blaschke product may have such a cluster point of poles, but does not roll off.)

In mixed sensitivity problems with irrational outer factors [13], it is also desirable to allow irrational weighting functions, which yields a problem equivalent to taking W_1 irrational in the present "pure" sensitivity case. Referring to (3.3), we see that for this example, W, M and F are all irrational. If these irrational functions were continuous and bounded from zero on the imaginary axis, there would be no need for further developments. However, irrational zeros require a different construction to prove sufficiency of (3.5), and the possibility of discontinuity of the outer factor (which does not occur in this beam example) makes it necessary to use a different condition than (3.5), since $W(z)$ will not be defined at a point of discontinuity. With this motivation, we shall consider the following generalization of Problem I:

PROBLEM G (general inner and outer): Assume, in addition to the assumptions of the Basic Problem, that $P(s) = \psi(s)\varphi(s)P_r(s)$, where

- $\psi(s)$ is a general inner function.
- $\varphi(s)$ and W_1 are general outer functions which have only finitely many "generalized zeros" (defined below) on the (extended) imaginary axis $j\mathbf{R}$. For technical reasons, we also assume that $\varphi(s)$ and W_1 are continuous at their own zeros on the (extended) imaginary axis.
- $P_r(s)$ is rational.

In order to give the necessary conditions which are the natural extension of (3.5), we define a *general zero* of f to be a point on the imaginary axis for which $|f|$ is not essentially bounded from 0 on any neighborhood of the point. We make this precise as follows:

Definition: Let f be a Lebesgue measurable function on $j\mathbf{R}$. For $z \in j\mathbf{R} \cup \{\infty\}$ and $\epsilon > 0$ take $N_\epsilon(z)$ to be the open interval $(z - i\epsilon,\ z + i\epsilon)$. (If $z = \infty$, take $N_\epsilon(z) = (\frac{1}{\epsilon}, \infty) \cup (-\infty, -\frac{1}{\epsilon})$.) Define

$$\operatorname{ess\,ran}(f, z) \triangleq \cap_{\epsilon>0} \{ \text{ essential range of } |f(x)| \text{ restricted to } N_\epsilon(z)\}$$

(the essential range of f at z). We say that z is a *general zero* of f if

$$\inf\ R_f(z) = 0,$$

and we write

$$\mathcal{V}(f) = \{z \in j\mathbf{R} \cup \{\infty\} | \inf\ R_f(z) = 0\}.$$

∎

Remark 4 *$\mathcal{V}(f)$ consists of zeros of finite order and certain singularities of f. It must have Lebesgue measure zero for f to be a boundary value function of a function in* $\mathbf{H}^\infty$.

EXAMPLE: $f(j\omega) = e^{-\sqrt{j\omega}}$ has a general zero at ∞ (taking the square root to have a branch cut on the negative real axis). $f(j\omega)$ is the boundary value of the $\mathbf{H}^\infty$ function $f(s) = e^{-\sqrt{s}}$, which is continuous on $j\mathbf{R}$ and has an essential singularity at ∞.

Remark 5 *In this third problem as well, as a consequence of our assumptions one can factor the plant as $P(s) = P_i(s) \cdot P_o(s) \cdot P_u(s)$ as in Remark 2.*

As indicated in Proposition 1, solutions to the first two problems (R and I via (3.4)) can proceed by first neglecting the factor $F(s)$ and solving the following:

$$\mu_i = \inf_{Q \in \mathbf{H}^\infty} \|W - MQ\|_\infty. \tag{3.7}$$

The transformation of (3.4) to (3.7) we call "outer factor absorption" because of the construction used to obtain an approximate solution to (3.4) from a solution to (3.7). This idea first appeared in [49, p. 591], where a construction was given for the case of P a rational function. For the case of P having rational outer and unstable parts but general inner part [10, p. 69] gave the first such construction. In the latter case it is also shown in [10],[18] that a sequence of approximate solutions $\{Q_n\}$ to (3.4) can be constructed from a solution to (3.7) such that they are rational functions.

Remark 6 *The infimum in (3.7) is actually a minimum [24, p. 135].*

Now we can proceed to develop parallel results for Problem G.

3.3 Results

We first state the necessary condition for outer factor absorption in Problem G.

Lemma 1 *Let $W \in \mathbf{L}^\infty(j\mathbf{R})$, let F be continuous at each point of $\mathcal{V}(F)$. Take*

$$\mu_o = \inf_{Q \in \mathbf{H}^\infty} \|W - FMQ\|_\infty. \tag{3.8}$$

Then

$$\mu_o \geq \sup\ R_W(z) \qquad \textit{for every} \quad z \in \mathcal{V}(F).$$

Proof: Let $z \in \mathcal{V}(F)$. For each $\epsilon > 0$ by continuity of F at z there is an interval $N_\epsilon = (z - j\gamma, z + j\gamma)$ with $\gamma > 0$ such that $|F(w)| < \epsilon$ for every $w \in N_\epsilon$. Take

$$\sigma = \sup R_W(z).$$

Suppose, contrary to the conclusion of the lemma,

$$\mu_o = \sigma - \alpha \text{ for some } \quad \alpha > 0, \tag{3.9}$$

and take $Q_\alpha \in \mathbf{H}^\infty$ to satisfy

$$\|W - FMQ_\alpha\|_\infty < \mu_o + \frac{\alpha}{4}.$$

Of course we always have

$$\inf_{Q} \|W - FMQ\|_\infty = \mu_o \leq \|W - FMQ_\alpha\|_\infty.$$

Take $\epsilon = \alpha \;/\; 8\|MQ_\alpha\|_\infty$. Then for $\omega \in N_\epsilon$,

$$\begin{aligned} |W(j\omega) - F(j\omega)M(j\omega)Q_\alpha(j\omega)| &\geq |W(j\omega)| - |F(j\omega)M(j\omega)Q_\alpha(j\omega)| \\ &> |W(j\omega)| - \frac{\alpha}{8} \qquad a.e. \end{aligned}$$

By definition of σ, there is a set of non-zero Lebesgue measure $S_\epsilon \subseteq N_\epsilon$ such that $|W(\omega)| \geq \sigma - \frac{\alpha}{8}$ for each $\omega \in S_\epsilon$. Thus for $\omega \in S_\epsilon$

$$\begin{aligned} \mu_o + \frac{\alpha}{4} &> \|W - FMQ_\alpha\|_\infty \\ &\geq |W(j\omega) - F(j\omega)M(j\omega)Q_\alpha(j\omega)| \qquad a.e. \\ &\geq \sigma - \frac{\alpha}{4} \qquad a.e., \end{aligned}$$

and we conclude that

$$\mu_o > \sigma - \frac{\alpha}{2}.$$

This contradiction with (3.9) establishes that $\mu_o \geq \sigma$. ∎

Remark 7 *When $W(j\omega)$ is continuous at z, $|W(z)| = \sup\ R_W(z)$.*

We next show that the conditions of Lemma 1 are also sufficient for outer factor absorption in Problem G. To motivate our development, we review the ideas behind the proof of sufficiency in Proposition 1 as presented in [10],[18] . The proof is by construction, which consists of two steps:

1. find a sequence $\{T_k\} \subset \mathbf{H}^\infty$ such that

$$\mu_k \to \mu_o \text{ as } \quad k \to \infty, \tag{3.10}$$

   where

$$\mu_k = \|W - T_k\|_\infty \;,$$

   and

$$T_k(z) = 0 \quad \text{for all} \quad z \in \mathcal{V}(F),$$

2. find a sequence f_n such that

   (a) $f_n = Fx_n$ for some $x_n \in \mathbf{H}^\infty$ (F divides f_n in $\mathbf{H}^\infty$)

   (b) $\|f_n\|_\infty \to 1$ as $n \to \infty$

   (c) $|1 - f_n| \to 0$ uniformly on finite intervals of the imaginary axis which exclude zeros of F.

We leave the details to the proof of Proposition 2 below, but here we give the ideas behind these two steps to motivate the proof.

Given a solution $\bar{Q}$ to (3.7), we would like to factor F out of it, and still have $F^{-1}\bar{Q} \in \mathrm{H}^\infty$ so that $F^{-1}\bar{Q}$ gives a solution to (3.8). However F^{-1} has poles at the points of $\mathcal{V}(F)$, and $\bar{Q}$ will not generally have zeros at these points, so this does not give a solution. Our approach is to find a sequence Q_n such that

$$\|W - FMQ_n\| \to \mu_o. \tag{3.11}$$

A naïve approach is multiply $\bar{Q}$ by another function so that the product is very close to $\bar{Q}$ over some union of intervals which excludes $\mathcal{V}(F)$, and then has zeros of the appropriate order at the points of $\mathcal{V}(F)$. The natural way to attempt this is the following: Assuming the desired zero is at ∞ and has order m, simply multiply $\bar{Q}$ by $\left(\frac{n}{s+n}\right)^m$. Letting $n \to \infty$, the interval on which the product is close to $\bar{Q}$ grows. This works for Problem R, but fails for Problem I.

The difficulty is as follows: In Problem I $(W - M\bar{Q})(s)$ has constant magnitude on the imaginary axis, and so $(W - M\bar{Q})(j\omega)$ describes a circle. In general, the locus will continue to revolve indefinitely as $|\omega|$ increases. Assuming for simplicity that $W(j\omega)$ approaches a constant as $|\omega| \to \infty$, we conclude that $M\bar{Q}(j\omega)$ approaches a circle with center $W(i\infty)$. Simply reducing the magnitude of $\bar{Q}$ (multiplying by the "roll-off" function $\left(\frac{n}{s+n}\right)^m$) as frequency increases without regard to phase, can result in the sensitivity locus leaving the asymptotic circle. For a roll-off function having only poles the minimal phase versus frequency characteristic is that of a one-pole roll-off. To obtain less phase but still get infinite attenuation at ∞ one must roll-off more slowly. That is the motivation for the roll-off functions h_n below which accomplish Step 1. This shows us how to modify $\bar{Q}$ so as to introduce zeros at the points of $\mathcal{V}(F)$ yet approach arbitrarily closely to μ_i.

The essential point is that in the rational plant case, at high frequency the product $P\bar{Q}$ approaches a real constant. Roll-off can therefore be introduced into the feedback loop at high frequency without regard for the phase, whereas in the case of general inner factors $P\bar{Q}$ has unbounded phase.

In order to factor F out, the zero we introduce by roll-off must be at least as steep as the zero of F. However Step 1 introduces only very slow roll off, with vanishing phase change as $n \to \infty$, in order to allow for the presence of general inner factors. As a consequence, we will have zeros of fractional order, and so the order of the zeros will still result in $F^{-1}T_n \notin \mathrm{H}^\infty$. The roll-off could be fast after a frequency at which the loop gain has decreased sufficiently. However one must be careful that the fast roll-off does not introduce excessive additional phase at lower frequencies.

The idea of Step 2 is that after the gain of T_n is sufficiently small, we can tolerate the introduction of large phase change by the roll-off function. The problem then becomes one of introducing the zero of appropriate order while constraining phase change to be small outside the interval where $\bar{Q}$ has not been rolled off sufficiently. In Problem I this can be accomplished using the same functions $\left(\frac{n}{s+n}\right)^m$ in the naïve approach above, but just taking the pole to be sufficiently large. In the more general case of irrational F, it is not necessarily obvious how to find a function which has zeros having order at least that of the zeros of

F (these latter zeros will generally be singularities), yet which is close to 1 on a specified interval excluding zeros. This construction for the irrational case, and the measure-theoretic arguments to accomodate general zeros, are the keys to the generalizations in this report.

Remark 8 *The sequence $\{Q_n\}$ satisfying (3.11) directly gives a sequence of stabilizing compensators via $C_n = \frac{Q_n}{1+PQ_n}$.*

For simplicity of exposition, we start with the case of a single irrational zero on the extended imaginary axis. Without loss of generality we assume this zero is at ∞. (Use a conformal map to move the zero to any finite point.) In parallel to [18], our construction here consists of two steps, which we state here as lemmas. Step 1 is accomplished using the following:

Lemma 2 *Suppose*

$$\mu_i = \inf_{Q \in \mathbf{H}^\infty} \|W - MQ\|_\infty = \|W - M\bar{Q}\|_\infty \tag{3.12}$$

with $\bar{Q} \in \mathbf{H}^\infty$, and

$$\mu_i \geq \sup\ R_W(\infty). \tag{3.13}$$

Let $h_n(s) \triangleq \left[\frac{\alpha}{s+\gamma}\right]^{1/n}$ with $\alpha > 0,\ \gamma > 0$. Then the sequence $T_n = h_n\bar{Q}$ satisfies $\mu_n \to \mu_i$ where $\mu_n \triangleq \|W - MT_n\|_\infty$.

Proof: See appendix. A simpler version which is suitable for treating Problem I is given in [10, pp. 69-71] and [18, pp. 519-521]. In these references it is shown that T_n can be taken to be rational functions. The parallel result is also true in the present case, but we do not prove it here. In Problem G we do not assume continuity of W at the zeros of P_o, and so the proof is more involved. ∎

Corollary 1 *Assume*

$$\{z_\ell\}_{\ell=1}^{m} \subset j\mathbf{R} \cup \{\infty\}.$$

Suppose (3.13) holds, and that $\bar{Q}$ solves (3.12). Let

$$h_n(s) \triangleq \prod_{j=1}^{m} \left[\alpha \cdot \frac{s - z_j}{s + \frac{\gamma+1}{\gamma - z_j}}\right]^{1/n} \quad \text{with} \quad \alpha > 0,\ \gamma > 0.$$

Then the sequence $T_n = h_n\bar{Q}$ satisfies $\mu_n \to \mu_i$ where $\mu_n \triangleq \|W - MT_n\|_\infty$.

Proof: The proof differs from that of Lemma 2 only in the details of the estimates. h_n here is the h_n function in Lemma 2 composed with conformal maps which take the points of $\mathcal{V}$ to ∞. ∎

For Step 2, in [10, pp. 69-71] and [18, pp. 521-522] we use the obvious construction for the rational case: we can easily find the order of the zero at infinity, and construct a

sequence satisfying the conditions in Step 2. For example, suppose that $F = \left(\frac{1}{s+1}\right)^m$. Then we can take $f_n = \left(\frac{n}{s+n}\right)^m$, and $x_n = F^{-1}f_n = \left(\frac{n(s+1)}{s+n}\right)^m$.

However, in the case of general irrational P it is not necessarily obvious how to construct a sequence $\{f_n\}$ to have sufficient roll-off at the irrational zeros on the imaginary axis as P, and yet approach 1 in the desired manner.

From Lemma 2, each T_n goes to zero at infinity, as desired, but the roll-off of the constructed zero at infinity will not generally be fast enough to make $F^{-1}T_k$ an element of H^∞. The issue addressed next is how to construct f_n (and x_n) so that the zero rolls off at least as fast as that of P, and yet have $Q_n = x_n T_n$ satisfy (3.11).

Lemma 3 *For any outer funct ɔn $p(s) \in \mathrm{H}^\infty$ there exists a sequence $\{x_n\}$ in H^∞ such that:*
(a) $\|px_n\| \to 1$ *as* $n \to \infty$,
and
(b) $|1 - p(j\omega)x_n(j\omega)| \to 0$ *uniformly a.e on compact subsets of the imaginary axis which exclude zeros of p(i.e., such that $R_p(j\omega)$ is bounded from 0 at every point on the interval).*

Proof: We need the following theorem [24, p. 85] which we quote here transformed from the unit disk to the imaginary axis and with a change of notation:

Theorem 2 *Let $p(s)$ be an outer function. Then there are functions $\{x_n\}$ in H^∞ such that*
$|x_n(s)p(s)| \leq 1$
$x_n(j\omega)p(j\omega) \to 1$ *almost everywhere.* ∎

We will not prove this theorem here, but we will use the fact that in the proof of this theorem, x_n is defined as follows: Let

$$u_n(j\omega) = \min(A_n,\ -\log|p(j\omega)|)$$

where each A_n, $0 < A_n < \infty$, is taken to be large enough so that

$$\sum_n \left(1 - p(1) \cdot \exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty} u_n(j\omega)\frac{d\omega}{1+\omega^2}\right]\right) < \infty.$$

One then takes

$$x_n(s) = \exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty} u_n(j\omega)\frac{\omega s + i}{\omega + is} \cdot \frac{1}{1+\omega^2}d\omega\right].$$

This $\{x_n\}$ satisfies the theorem.

This theorem directly gives us (a), and for (b) we need only show the uniformity of the convergence $|1 - f_n| \to 0$ on finite intervals of the imaginary axis which exclude zeros of p, with $f_n = x_n \cdot p$. For this, we check that on any closed finite interval $\Omega \subset j\mathbf{R}$ such that $\mathcal{V} \cap \Omega = \emptyset$, for n sufficiently large, $|f_n| = 1$ and $\arg(f_n) < \frac{1}{n}$ *a.e.* on the interval.

Let $S_n = \{s | s \in j\mathbf{R} \text{ and } n \leq -\log|p(s)|\}$. Then $S_{n+1} \subset S_n$, and we take $\bar{S}_n \triangleq \text{closure}(S_n \setminus (\Omega \cap S_n))$. Using the assumption that R_p is bounded from 0 on Ω, we can take n_Ω sufficiently large so that for every $n \geq n_\Omega$, $\text{meas}(\Omega \cap S_n) = 0$ and $\Omega \cap \bar{S}_n = \emptyset$. Then for $s \in \Omega$,

$$\begin{aligned} |f_n(s)| &= |F(s)| \cdot |x_n(s)| \\ &= |F(s)| \cdot \exp[u_n(s)] \\ &= 1 \quad a.e. \end{aligned}$$

The phase of f_n is given a.e. on the imaginary axis by:

$$\begin{aligned} \arg(f_n(j\omega)) &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{t}{1+t^2}+\frac{1}{\omega-t}\right)\log|f_n(it)|dt \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\omega t+1}{(1+t^2)(\omega-t)}\right)\log|f_n(it)|dt. \end{aligned}$$

Then

$$\arg(f_n(j\omega)) = -\frac{1}{\pi}\int_{S_n}\left(\frac{\omega t+1}{(1+t^2)(\omega-t)}\right)\log|f_n(it)|dt \qquad a.e.$$

If $\omega \in \Omega$, we can write a.e.

$$\begin{aligned} |\arg(f_n(j\omega))| &\leq \frac{1}{\pi}\left(\operatorname*{ess\,sup}_{t\in S_n}\left|\frac{\omega t+1}{\omega-t}\right|\right)\int_{S_n}\left|\frac{\log|f_n(it)|}{1+t^2}\right|dt \\ &\leq \frac{1}{\pi}\left(\operatorname*{ess\,sup}_{t\in S_n,\ \nu\in\Omega}\left|\frac{\nu t+1}{\nu-t}\right|\right)\int_{S_n}\left|\frac{\log|f_n(it)|}{1+t^2}\right|dt. \end{aligned}$$

Now, $\frac{\log|f_n(it)|}{1+t^2} \in L^1$ because $f_n \in \mathbf{H}^\infty$ [24, p. 66], and Ω and $\bar{S}_n$ are disjoint by choice of n, so the *ess sup* is finite. This provides a uniform bound on $|\arg(f_n(j\omega))|$ for $\omega \in \Omega$. Also, $\int_{S_n}\left|\frac{\log|f_n(it)|}{1+t^2}\right|dt \to 0$ as $n \to \infty$, since $\textit{meas}(S_n) \to 0$ and $|\log|f_n(it)||$ is non-increasing as a function of n for fixed t. Finally, $\sup_{t\in S_n,\ \nu\in\Omega}\left|\frac{\nu t+1}{\nu-t}\right|$ is non-increasing as n increases. Thus, as $n \to \infty$, $\arg(f_n) \to 0$ uniformly on any closed interval which excludes zeros of $|p|$. ∎

Remark 9 *The key step is separating S_n from Ω with closed sets, except for a set of measure zero, so that the essential distance between the two is bounded from zero. This is what allows us to bound the phase of f_n.*

Now we can prove the following:

Proposition 2 *Let*

$$\mu_o = \inf_{Q\in\mathbf{H}^\infty}\|W - FMQ\|_\infty ,$$

where W, F and $M \in \mathbf{H}^\infty$, and F is outer and M is inner. Let

$$\mu_i = \inf_{Q \in \mathbf{H}^\infty} \|W - MQ\|_\infty .$$

Assume that $\mathcal{V}(F)$ is finite, and that F is continuous at each point of $\mathcal{V}(F)$. Then

$$\mu_i \geq \sup \ R_W(z) \text{ for all } \ z \in \mathcal{V}(F), \tag{3.14}$$

if and only if

$$\mu_o = \mu_i . \tag{3.15}$$

Proof: By Lemma 1 (3.14) is a necessary condition for (3.15). For the converse, it is obvious that $\mu_o \geq \mu_i$. We shall construct a sequence Q_n such that $\mu_o \leq \|W - FMQ_n\| \to \mu_i$, and therefore conclude that $\mu_o \leq \mu_i$. Suppose first that $\mathcal{V}(F) = \{\infty\}$. Let $W_\infty = \sup \ R_W(\infty)$, and assume $\mu_i \geq W_\infty$. Take $h_n(s) \triangleq \left[\frac{\alpha}{s+\gamma}\right]^{1/n}$ with $\alpha > 0$, $\gamma > 0$ and $T_n = h_n\bar{Q}$ as in Lemma 2, where $\bar{Q}$ solves (9) . Let $\{x_n\}$ be the sequence resulting from substituting F for p in Lemma 3, and we take $Q_n = x_n T_n$. We shall show that $\|W - FMQ_n\| \to \mu_i$ by finding conditions on n so that

$$\|W - FMQ_n\| < \mu_i + \epsilon. \tag{3.16}$$

For $\epsilon > 0$, for each n we claim we can take $\omega_n > 0$ to satisfy for $|\omega| > \omega_n$ both

$$|W(j\omega)| < W_\infty + \epsilon \qquad a.e. \tag{3.17}$$

and

$$|h_n(j\omega)| < \frac{\mu_i - |W(j\omega)| + \epsilon}{\mu_i + |W(j\omega)|} \ a.e. \tag{3.18}$$

To verify this claim, note that (3.17) will hold by definition of W_∞ and by hypothesis on μ_i for $|\omega|$ sufficiently large independently of n. For (3.18) to hoid it is easy to check that a sufficient condition is

$$|\omega_n| \geq \alpha \left(\left[\frac{\mu_i + |W(j\omega)|}{\mu_i - |W(j\omega)| + \epsilon} \right]^n - \frac{\gamma}{\alpha} \right) \qquad a.e. \text{ on } \ (\omega_n, \infty).$$

This will always hold for large enough n because $|W(j\omega)|$ is essentially bounded at ∞, and $|W(j\omega)| - \epsilon$ is essentially bounded from μ_i at ∞.

Let $\{f_n\}$ be a subs-quence of $\{Fx_n\}$ such that

$$|1 - f_n(j\omega)| < \frac{1}{n^2} a.e. \text{ for } |\omega| \leq \omega_n. \tag{3.19}$$

The way we pick n will be to pick it large enough so that for $|\omega| \leq \omega_n$(13) holds. This works because for $|\omega| > \omega_n$, our definition of ω_n will ensure (13) holds. We see this as follows:

We know that $\bar{Q}$ satisfies the condition that $(W - P_i\bar{Q})$ is all-pass, say

$$\mu_i e^{i\alpha(\omega)} = (W - M\bar{Q})(j\omega) a.e. \tag{3.20}$$

Recalling the definition $X_n(j\omega) \triangleq W(j\omega) - P_i(j\omega)T_n(j\omega)$, assuming we have chosen ω_n to satisfy (3.17) and (3.18), and taking μ_i and $\alpha(\omega)$ as in (3.20), we have

$$\mu_i - |W(j\omega)| + \epsilon > |h_n(j\omega)| \cdot (\mu_i + |W(j\omega)|) a.e.$$

$|f_n(j\omega)| \leq 1$ *a.e.*, so

$$\begin{aligned} |f_n(j\omega)h_n(j\omega)(\mu_i e^{i\alpha(\omega)} - W(j\omega)| &\leq |h_n(j\omega)| \cdot |\mu_i e^{i\alpha(\omega)} - W(j\omega)| \\ &\leq |h_n(j\omega)|(\mu_i + |W(j\omega)|) \\ &< \mu_i - |W(j\omega)| + \epsilon \quad a.e. \end{aligned}$$

But

$$(W(j\omega) - F(j\omega)M(j\omega)x_n(j\omega)T_n(j\omega)) = W(j\omega) + f_n(j\omega)h_n(j\omega)(\mu_i e^{i\alpha(\omega)} - W(j\omega)) \quad a.e.,$$

so

$$\begin{aligned} |(W(j\omega) - F(j\omega)M(j\omega)x_n(j\omega)T_n(j\omega))| &\leq |W(j\omega)| + |f_n(j\omega)h_n(j\omega)(\mu_i e^{i\alpha(\omega)} - W(j\omega))| \\ &< \mu_i + \epsilon \quad a.e. \end{aligned}$$

So now we treat the case $|\omega| \leq \omega_n$, essentially repeating the argument in Lemma 2. Let

$$Y_n(j\omega) \triangleq W(j\omega) - M(j\omega)f_n(j\omega)T_n(j\omega)$$

and

$$g_n(j\omega) \triangleq f_n(j\omega)h_n(j\omega).$$

$$|Y_n(j\omega)|^2 = |W(j\omega) + g_n(j\omega)[\mu_i e^{i\alpha(\omega)} - W(j\omega)]|^2 a.e.$$

Set $\omega_m \in [0, \omega_n]$ to a frequency at which

$$\left(\operatorname*{ess\ sup}_{\omega \in [0,\omega_n]} |Y_n(j\omega)| \right) \in R_Y(\omega_m).$$

Now let $\gamma = \arg(g_n)$ and define

$$g = \sup\ R_{g_n}(\omega_m),\ \delta = \sup\ R_\gamma(\omega_m),\ W_m = \sup\ R_W(\omega_m), \text{ and } \alpha = \alpha(\omega_m).$$

These are functions of n, as is ω_m. One can see from (3.19) that $|\arg[f_n(j\omega_m)]| < 1/n^2$, and from the definition of $h_n(s)$ that $0 \leq \arg[h_n(j\omega_m)] \leq 2\pi/n$. Therefore $-1/n^2 < \delta < 2\pi/n + 1/n^2$.

Now we show that we can pick n as a function of $\epsilon > 0$ such that $\|Y_n\| \leq \mu_i + \epsilon$. As in the proof of Lemma 2, $|Y_n(j\omega)|^2 \leq [W_m(1-g) + g\mu_i]^2 + 2g\mu_i W_m \delta$ *a.e.* Given n, there are two possibilities:

case (i). ($\omega_m > n$) Exactly as in the proof of Lemma 2 we find $|Y_n(j\omega)|^2 \leq \mu_i^2 + \epsilon$ *a.e.* for sufficiently large n.

case (ii). ($\omega_m \leq n$) Then since $\|W(j\omega)\|_\infty \geq W_m$ and $g \leq 1$, as before we have

$$(\sup R_Y(\omega_m))^2 \leq \mu_i^2 + (1-g) \cdot 2\|W(j\omega)\|_\infty \mu_i + \frac{4\pi\mu_i\|W(j\omega)\|_\infty}{n} + (1-g)^2\|W(j\omega)\|_\infty^2 \quad a.e.$$

and

$$1 - g \leq |1 - g_n(j\omega_m)| = |1 - f_n(j\omega_m)h_n(j\omega_m)|$$

Using the estimate of $(1-h)$ from the proof of Lemma 2 along with (3.18) we find $1 - g \to 0$, and we conclude that $\mu_i - \|X_n\|_\infty \to 0$ as $n \to \infty$.

To handle the case of $\mathcal{V}(F)$ finite with more than one point, note that Corollary 1 and Lemma 3 already apply to this case. Use the above to find conditions on n for each element of $\mathcal{V}$ after transforming it to ∞, as in Corollary 1, and take the maximum of the resulting values. ∎

This proposition extends readily to the case of mixed sensitivity, which amounts to the following problem [14]: Solve

$$\mu \ \triangleq \ \inf_{Z\in\mathbf{H}^\infty} \left\| \begin{bmatrix} W - FMZ \\ V \end{bmatrix} \right\|_\infty . \tag{3.21}$$

with

$$W(s) \triangleq W_1^*(s)W_1(s)W_2^*(s)^{-1}R^*(s) - W_2(s)R(s)^{-1}N(s)X(s),$$
$$M(s) \triangleq N_i(s)D_i(s),$$
$$F(s) \triangleq N_o(s)D_o(s)W_2(s)R^{-1}(s),$$
$$V(s) \triangleq W_1(s)R(s),$$

and $R \in \mathbf{H}^\infty$ and outer satisfies

$$\frac{W_2^*(s)W_2(s)}{W_1^*(s)W_1(s) + W_2^*(s)W_2(s)} = R^*(s)R(s).$$

Here $N_i, D_i \in \mathbf{H}^\infty$ are inner, N_o, D_o and $W_1 \in \mathbf{H}^\infty$ are outer, and W_2 is outer (but not necessarily in $\mathbf{H}^\infty$).

Corollary 2 *Let F, M, W, and V be as above. Assume W_1 is continuous $j\mathbf{R}$, and F is continuous on $\mathcal{V}(F)$, which has finite cardinality. Define*

$$\mu_i \triangleq \inf_{Z\in\mathbf{H}^\infty} \left\| \begin{bmatrix} W - MZ \\ V \end{bmatrix} \right\|_\infty .$$

Assume further that

$$\mu_i > \sup\{\|V\|_\infty, |W_1(j\omega)| : j\omega \in \sigma_e(M)\}. \tag{3.22}$$

Take

$$\mu_o \triangleq \inf_{Z\in \mathbf{H}^\infty} \left\| \begin{bmatrix} W - MFZ \\ V \end{bmatrix} \right\|_\infty .$$

Then

$$\mu_o = \mu_i$$

if and only if

$$\mu_i^2 \geq \sup_{z\in\mathcal{V}(F)} R_{W^*W+V^*V}(z).$$

Proof: The assumption (3.22) allows one to transform the mixed sensitivity problem to a sensitivity problem [44]. The calculation is as follows: For $Z \in \mathbf{H}^\infty$

$$\mu_i^2 \geq (W(j\omega) - M(j\omega)Z(j\omega))^*(W(j\omega) - M(j\omega)Z(j\omega)) + V^*(j\omega)V(j\omega) \quad a.e.$$

From (3.22) $\mu_i^2 - V^*(j\omega)V(j\omega) > 0$, so we can find an outer spectral factor T such that

$$T^*(j\omega)T(j\omega) = \mu_i^2 - V^*(j\omega)V(j\omega).$$

Then we have

$$1 \geq |T^{-1}(j\omega)(W(j\omega) - M(j\omega)Z(j\omega))|.$$

Of course, we must have

$$1 = \inf_{Z\in\mathbf{H}^\infty} \|T^{-1}(W - MZ)\|_\infty$$

$$= \inf_{Z\in\mathbf{H}^\infty} \|T^{-1}W - MZ\|_\infty$$

Under the assumption (3.22) it follows that there exists a unique $\bar{Z} \in \mathbf{H}^\infty$ satisfying

$$1 = |T^{-1}W - M\bar{Z}| \qquad a.e.$$

From Proposition (2) we have that the condition

$$1 \geq \sup_{z\in\mathcal{V}(F)} R_{W^{-1}T}(z)$$

is necessary and sufficient for

$$1 = \inf_{Z\in\mathbf{H}^\infty} \|T^{-1}W - MFZ\|_\infty.$$

But we know then that for every $\epsilon > 0$ we can find $Z_\epsilon \in \mathbf{H}^\infty$ such that

$$1 + \epsilon > |T^{-1}(j\omega)(W(j\omega) - M(j\omega)Z_\epsilon(j\omega))| \quad a.e.$$

or

$$(1+\epsilon)|T(j\omega)| > |W(j\omega) - M(j\omega)Z_\epsilon(j\omega)| \quad a.e.$$

which implies

$$\|T\| \geq \inf_{Z\in\mathbf{H}^\infty} \|W - MZ\|.$$

Thus, after some more algebra, we obtain the desired conclusion. ∎

The assumption (3.22) corresponds to the case in which the norm of the corresponding "Hankel plus Toeplitz" operator [50], [14] corresponds to an eigenvalue.

3.4 Conclusions

We have established the validity of "outer factor absorption" as a step in the solution of $\mathbf{H}^\infty$ sensitivity and mixed sensitivity problems for a wide class of distributed parameter plants. In the process, we have presented a constructive approach for "re-inserting" the outer factor into, or "extracting" it from, a solution. Although we know of no physical plants not covered by the class treated, it is far from general: The reader may wish to consider such pathological cases as W and F both being discontinuous at a zero of F, F a continuous function with zeros dense a some point, or F having zeros dense on some segment of the imaginary axis.

Chapter 4

SISO Mixed Sensitivity

4.1 Introduction

The objective in this chapter is to solve as explicitly as possible the $\mathbf{H}^\infty$-infimal mixed sensitivity problem for linear time-invariant single/input - single/output distributed parameter systems.

Our motivation for considering distributed parameter $\mathbf{H}^\infty$ problems comes from work intended to address infinite dimensional models (irrational transfer function) of large space structures. The model motivating our departure from existing results is the damped flexible beam of Section 2.1.

In all previous work (*e.g.* [9],[10],[19],[51],[50],[34]), the authors assume rational proper weighting functions, plant outer factors which are invertible, and finite dimensional instabilities. Our assumptions here are more general in each regard, although in the mixed sensitivity computations we do not treat irrational weighting functions.

[50] assumes certain operators to be continuous on the imaginary axis, and this assumption was used to compute the essential spectrum of an operator. In the unstable plant case with general inner factor this is no longer true. In this report we show that the operator arising from the possibly discontinuous function differs from continuous operator by a compact operator, and therefore has the same essential spectrum. This idea is really an application of the same technique which has been used in the distributed-plant $\mathbf{H}^\infty$ problem from the start [9], [19], [10].

At one point, computations of certain basis functions become complicated by the presence of irrational functions, and the dimension of a related subspace is increased by the introduction of an extra finite Blaschke product. Below we present calculations treating this case.

Finally, we present the application of these calculations to the model of a damped flexible beam of Section 2.1. This model has an irrational and non-invertible outer factor, and this fact has motivated us to develop the general solution for that case.

We would like to mention that the paper [34] treats a similar problem, although only rational outer factors are allowed there. Even in that rational case, we believe that the present work indicates the connection with previous results in a more transparent manner.

4.2 The Mixed Sensitivity Problem

The general setup is indicated in the block diagram below.

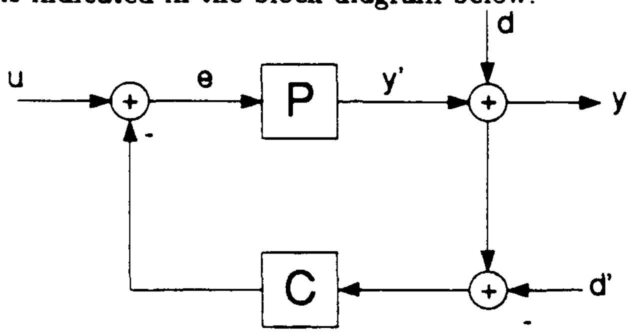

Figure 4.1: General Feedback System

The *sensitivity* $S(s)$ is the transfer function from the disturbance $d(s)$ to the output $y(s)$. The *complementary sensitivity* $1 - S(s)$ is the transfer function from the disturbance $d'(s)$ to $y(s)$, which is the same as the transfer function from $d(s)$ to $y'(s)$. Obviously a small sensitivity over a frequency range of interest means good disturbance rejection, and it is known that a small complementary sensitivity means good stability margin. [6], [40].

The $\mathbf{H}^\infty$ *mixed sensitivity* problem is the following [32]: find a stabilizing feedback compensator which minimizes

$$\sup_\omega \left(|W_1(j\omega)S(j\omega)|^2 + |W_2(j\omega)[1 - S(j\omega)]|^2\right).$$

where $W_1(s)$ and $W_2(s)$ are frequency-dependent weighting functions which serve to emphasize or deemphasize the importance we attach to the magnitude of S or $(1 - S)$ at different frequencies. This problem is equivalent to the following: choose a stabilizing feedback compensator $C(s)$ to solve

$$\inf_{C(s)} \|T(s)\|_\infty, \tag{4.1}$$

where we regard

$$T(s) = \begin{bmatrix} W_1(s)S(s) \\ W_2(s)(1 - S(s)) \end{bmatrix}$$

as an operator from the Hardy space $\mathbf{H}^2$ to $\mathbf{H}^2 \times \mathbf{H}^2$.

4.2.1 Assumptions about the Plant

We first describe the structure of our plant model. We assume that the plant has transfer function $P(s)$, with the following further assumptions: (See [13] and [12] for further explanation.)

Assumption A: $P(s)$ has a coprime factorization $P(s) = N(s)/D(s)$ satisfying

$$N(s)X(s) + D(s)Y(s) = 1, \tag{4.2}$$

where $N(s)$, $D(s)$, $X(s)$ and $Y(s)$ are all functions in $\mathbf{H}^\infty$.

Assumption B: Let $N(s) = N_i(s)N_o(s)$ and $D(s) = D_i(s)D_o(s)$ be the inner-outer factorizations of these functions. We assume:

B.1: $|D_o(s)|$ is bounded away from 0 at ∞ in the right half plane $Re(s) > 0$.

B.2: D_i is a finite Blaschke product.

B.3: Let S_o denote the subset of the extended imaginary axis $i\mathbf{R} \cup \{\infty\}$ on which $W_2N_oD_o$ vanishes essentially. Then $W_2N_oD_o$ is continuous at each point of S_o.

Assumption C:

C.1: $W_1(s) \in \mathbf{H}^\infty \cap \mathbf{C}$ is outer;

C.2: $W_1(s)W_2^{-1}(s) \in \mathbf{H}^\infty$, $W_2(s)N(s)X(s) \in \mathbf{H}^\infty$;

C.3: $W_1(s)$ and $W_2(s)$ are rational functions of s.

Remark 10 *It seems to have been known that stabilizability is equivalent to the existence of a coprime factorization (see, for example, [Zames and Francis P. 590]), but the first published demonstrations of this, for the case of plants which are ratios of* $\mathbf{H}^\infty$ *functions, appear to be [28] and [42]. However the plants considered by these authors do not further restrict the denominators, and so non-causal plants are allowed. As a consequence of this setup , non-causal systems may arise as "stabilizing" compensators. Our assumptions B.1 and B.2 eliminate this possibility.*

Remark 11 *Assumptions B.1 and B.2 together guarantee that $P(s)$ is analytic and bounded in some right half plane. Thus $P(s)$ is a shifted version of an element of* $\mathbf{H}^\infty$*, and so is the Laplace transform of a growing exponential times the (distributional) impulse response of a causal stable linear time invariant system. Assumption B.2 is not necessary for this (we could have simply assumed $D(s)$ is bounded away from* 0 *at* ∞ *in the right half plane), but it is satisfied by all physically motivated problems we have seen.*

Remark 12 *Assumption B.3 we use to prove that we really can "absorb N_oD_o into the free parameter," as we explain below. (We shall also need some assumptions on the weighting functions.)*

Remark 13 *We know of no plants arising from models of real problems which violate any of these assumptions. In the introduction we presented an example consisting of a damped flexible beam which has irrational N_o.*

Remark 14 *We use Assumption C.3 to prove Lemma 4 in Section 4.4.2 and successive results. We would like to eliminate this assumption for the reason alluded to in Section 2.2: in order to obtain a proper compensator by using an improper weighting function W_2, Assumption C.2 would be sufficient. However, our proof of Lemma 4 requires that the function $(W_1W_2/W)^*$ be meromorphic in the right half plane, which will not hold for some irrational W_2 which satisfy C.2. C.1 and C.2 imply W_2^{-1} is analytic in $\mathbf{C}^+$ and outer.)*

Remark 15 *We allow $W_2(s)$ to be an improper function. In practice one would pick W_2 based upon one's knowledge of $P(s)$. From this perspective, we are allowing W_2 to have poles at the zeros of $P(s)$ on the imaginary axis, if the designer wishes. Actually, we not only allow W_2 to be improper, we recommend it for the following reason: If W_2P is not strictly proper, then when we find a solution to the mixed sensitivity problem, the optimal complementary sensitivity will be strictly proper, in fact enough so to make the resulting compensator proper. This can also be accomplished as in [44] by considering $W_2P^{-1}(1-S)$ instead of $W_2(1-S)$, but the latter technique is less general. (Our assumptions encompass the case of [44] because [44] assume W_2 and P both have no poles or zeros on the imaginary axis.)*

4.2.2 Transformation to Standard Form

In this section we review results from [13] which show how to find an operator

$$T_1 : \mathbf{H}^2 \to \mathbf{H}^2 \times \mathbf{H}^2$$

having structure

$$T_1(s) \triangleq \begin{bmatrix} G(s) - \overline{M}(s)Z(s) \\ F(s) \end{bmatrix}$$

such that $T^*(s)T(s) = T_1^*(s)T_1(s)$. These calculations may be considered a clarification and extension of the calculations in [8], where the assumptions about the plant and weighting functions were less general, and the results slightly less explicit.

Proposition 3 *We can write*

$$\frac{W_2^*(s)W_2(s)}{W_1^*(s)W_1(s) + W_2^*(s)W_2(s)} = R^*(s)R(s) \tag{4.3}$$

with $R \in \mathbf{H}^\infty$ and outer. Furthermore, $R^{-1} \in \mathbf{H}^\infty$.

Proof: Obviously $\frac{W_1^*(j\omega)W_1(j\omega)}{W_2^*(j\omega)W_2(j\omega)}$ is non-negative. Thus

$$\frac{W_2^*(j\omega)W_2(j\omega)}{W_1^*(j\omega)W_1(j\omega) + W_2^*(j\omega)W_2(j\omega)} = \left(1 + \frac{W_1^*(j\omega)W_1(j\omega)}{W_2^*(j\omega)W_2(j\omega)}\right)^{-1} \in \mathbf{L}^\infty(\mathbf{R}).$$

Now from [Hoffman, *p.* 53] (translated from $\mathbf{H}^\infty$ of the disk to the half-plane) we know then that a necessary and sufficient condition to factor a function $f(j\omega) \in \mathbf{L}^\infty(\mathbf{R})$ with $f \geq 0$ *a.e.* as $f = h^*h$, with $h \in \mathbf{H}^\infty$, is that $\log[f(j\omega)]/(1+\omega^2) \in \mathbf{L}^1(\mathbf{R})$. So we check:

$$\left|\log\left(\frac{W_2^*(j\omega)W_2(j\omega)}{W_1^*(j\omega)W_1(j\omega) + W_2^*(j\omega)W_2(j\omega)}\right)\right| = \log\left(1 + \frac{W_1^*(j\omega)W_1(j\omega)}{W_2^*(j\omega)W_2(j\omega)}\right)$$

$$\leq \log\left(1 + \left\|W_1(j\omega)W_2^{-1}(j\omega)\right\|_\infty^2\right).$$

This gives (4.3) by using the result cited from [26]. Taking an inner- outer factorization, it is obvious that R can always be chosen to be outer. That $R^{-1} \in \mathbf{H}^\infty$ follows from $\left(\frac{W_2^*(j\omega)W_2(j\omega)}{W_1^*(j\omega)W_1(j\omega)+W_2^*(j\omega)W_2(j\omega)}\right)^{-1} \in \mathbf{L}^\infty$. ∎

Remark 16 *$R(s) \in \mathbf{H}^\infty$ is outer since $R^{-1} \in \mathbf{H}^\infty$.*

Remark 17 *This result does not require Assumption C.3, i.e., rationality of the weighting functions is not required for the factorization in (4.3).*

Now taking

$$W(s) \triangleq W_2(s)R^{-1}(s) \tag{4.4}$$

$$G(s) \triangleq W_1^*(s)W_1(s)W^*(s)^{-1} - W(s)N(s)X(s), \tag{4.5}$$

$$\overline{M}(s) \triangleq N(s)D(s)W_2(s)R^{-1}(s), \tag{4.6}$$

and

$$F(s) \triangleq W_1(s)R(s), \tag{4.7}$$

it is easy to check that

$$T^*(s)T(s) = (G(s) - \overline{M}(s)Z(s))^*(G(s) - \overline{M}(s)Z(s)) + F^*(s)F(s).$$

Since $\|T(s)\|_\infty^2 = \|T^*(s)T(s)\|_\infty$, we can find $\|T\|_\infty$ by finding

$$\left\| \begin{bmatrix} G - \overline{M}Z \\ F \end{bmatrix} \right\|_\infty .$$

Thus

$$\mu \triangleq \inf_{Z \in \mathbf{H}^\infty} \left\| \begin{bmatrix} G - \overline{M}Z \\ F \end{bmatrix} \right\|_\infty \tag{4.8}$$

$$= \inf_{Z \in \mathbf{H}^\infty} \left\| \begin{bmatrix} G - N_i N_o D_i D_o W_2 R^{-1} Z \\ F \end{bmatrix} \right\|_\infty . \tag{4.9}$$

R is invertible in $\mathbf{H}^\infty$, so we can immediately write ("absorbing" R^{-1} into the free parameter Z)

$$\mu = \inf_{Z \in \mathbf{H}^\infty} \left\| \begin{bmatrix} G - N_i N_o D_i D_o W_2 Z \\ F \end{bmatrix} \right\|_\infty . \tag{4.10}$$

4.2.3 Absorption of the Outer Factor

To reduce (4.10) to the desired form, we need Proposition 2 of Chapter 3. The point is that we can absorb the outer factor of NDW_2 into the "free parameter", just as in the rational [49] and rational-with-delay cases for pure sensitivity [10], [18].

The following is simply a translation of Proposition 2 of Chapter 3 to the present context:

Corollary 3 *Let μ be as in (4.8) above, and take*

$$\mu_i = \inf_{Z \in \mathbf{H}^\infty} \left\| \begin{bmatrix} G - N_i D_i Z \\ F \end{bmatrix} \right\|_\infty .$$

Suppose $\mathcal{V}(N_o D_o W_2)$ is a finite set consisting only of ordinary zeros. Then $\mu = \mu_i$ if and only if

$$\mu_i^2 \geq \sup \bigcup_{z \in \mathcal{V}(N_o D_o W_2)} \operatorname{ess\,ran}(G^* G + F^* F, z) \tag{4.11}$$

is satisfied. ∎

Remark 18 *$\mathcal{V}(N_o D_o W_2)$ is the set S_o defined in Assumption B.3.*

Remark 19 *It is obviously the case that $\mu_i \leq \mu$, so this proposition asserts necessary and sufficient conditions for $\mu \leq \mu_i$ when S_o contains only ordinary zeros.*

4.3 Implications for Design

In the case of stable plants, one can take $X = 0$ in (4.2), and assuming the outer factor can be absorbed into the parameter "Z" as described above, we see that the infimal mixed sensitivity is independent of the outer factor of the plant.

In greater generality, suppose $D_i = 1$. This means that any unstable poles of the plant are on the imaginary axis. Then using Corollary 3, Proposition 2 and the fact that $R^{-1}X \in \mathrm{H}^\infty$, we have

$$\begin{aligned}
\mu &= \inf_{Z \in \mathbf{H}^\infty} \left\| \begin{bmatrix} W_1^* W_1 (W_2^*)^{-1} R^* - W_2 R^{-1} N X - N D W_2 Z \\ W_1 R \end{bmatrix} \right\|_\infty \\
&= \inf_{Z \in \mathbf{H}^\infty} \left\| \begin{bmatrix} W_1^* W_1 (W_2^*)^{-1} R^* - W_2 R^{-1} N X - N_i Z \\ W_1 R \end{bmatrix} \right\|_\infty \\
&= \inf_{Z \in \mathbf{H}^\infty} \left\| \begin{bmatrix} W_1^* W_1 (W_2^*)^{-1} R^* - W_2 R^{-1} N X - W_2 N_o N_i Z \\ W_1 R \end{bmatrix} \right\|_\infty \\
&= \inf_{Z \in \mathbf{H}^\infty} \left\| \begin{bmatrix} W_1^* W_1 (W_2^*)^{-1} R^* - W_2 N (R^{-1} X - Z) \\ W_1 R \end{bmatrix} \right\|_\infty \\
&= \inf_{Z \in \mathbf{H}^\infty} \left\| \begin{bmatrix} W_1^* W_1 (W_2^*)^{-1} R^* - W_2 N Z \\ W_1 R \end{bmatrix} \right\|_\infty \\
&= \inf_{Z \in \mathbf{H}^\infty} \left\| \begin{bmatrix} W_1^* W_1 (W_2^*)^{-1} R^* - N_i Z \\ W_1 R \end{bmatrix} \right\|_\infty
\end{aligned}$$

Thus the infimal mixed sensitivity is independent of the outer factor of the plant, N_o/D_o.

An interesting application of this observation in the stable plant case is an explanation of some previous numerical experiments. In the "Integrated Structural Analysis and Control"

project at Aerospace Corp. [1], the goal was to produce a computational methodology for improving the performance of large space structures by simultaneously designing the structure and control systems. We used a simple idealized satellite example, where we imposed mass and cross-sectional area constraints on structural members and a complementary sensitivity constraint to provide stability margins, and we applied a general-purpose non-linear programming package to minimize the weighted sensitivity subject to these constraints.

In this case, we found there to be essentially no benefit in simultaneous design over separately computed structural and control designs. In [29] we presented the hypothesis that this "separability" was due to the absence of right half plane zeros in the plant model. The conjecture was heuristically justified by appealing to the well-known fact that for the pure sensitivity problem it is the inner part of the plant which limits performance. Thus the idea was that whatever the structure was, so long as the plant had a trivial inner factor, the limiting norm of the sensitivity transfer function did not depend upon the plant, and amounted to whatever the complementary sensitivity constraint dictated. Previously, it was not clear why the limit which the complementary sensitivity constraint imposed should have been independent of the plant. The present result provides the explanation, since for the problem considered, the structural design constraints present preserved the lack of inner and unstable factors.

4.4 Optimal H^∞ Mixed Sensitivity

We employ Corollary 3 by making the additional assumption

Assumption D: $\mu_i \geq \sup \bigcup_{z \in S_0} \operatorname{ess\,ran}(G^*G + F^*F, z)$.

Accordingly, we shall treat the problem

$$\mu = \min_{Z \in H^\infty} \| \begin{pmatrix} G - MZ \\ F \end{pmatrix} \|_\infty \tag{4.12}$$

where R, G, and F are as in (4.3), (4.5) and (4.7), and M is given by

$$M \triangleq N_i D_i. \tag{4.13}$$

Let

$$\mathcal{T} \triangleq (\Pi_+ M G^* \Pi_- M^* G + \Pi_+ F^* F) : \mathbf{H}^2 \to \mathbf{H}^2.$$

As pointed out in [44], $\mu^2 = \|\mathcal{T}\|$. This is the starting point for the development in [50].

Now we parallel the developments in [50]. The basic idea is that if $\mathcal{T}$ has an eigenvalue larger than the essential spectral radius of $\mathcal{T}$, then $\|\mathcal{T}\|$ is equal to the largest eigenvalue of $\mathcal{T}$. So one computes the esssential spectral radius, and searches for the largest eigenvalue. Since

$$\|\mathcal{T}\| \leq \left\| \begin{pmatrix} G \\ F \end{pmatrix} \right\|_\infty$$

we know an upper bound for the largest eigenvalue. Therefore a search can be confined to a known finite interval.

There are two complications in applying [50] directly. First, in [50] the operators G and F were assumed to be continuous on the imaginary axis. This allowed the authors of [50] to apply a theorem in [33, p. 125] to the continuous (on $j\mathbf{R}$) function $G^*G + F^*F$ to determine the essential spectrum of $\Pi_{\mathbf{K}}(G^*G + F^*F)|_{\mathbf{K}}$ (the compression of $G^*G + F^*F$ to $\mathbf{K}$), where $\mathbf{K} \triangleq \mathbf{H}^2 \ominus M\mathbf{H}^2$ (after showing that $\mathcal{T}$ differs from $\Pi_{\mathbf{K}}(G^*G + F^*F)|_{\mathbf{K}}$ by a compact operator). In the present work G is not continuous (in the general, unstable case), due (at least) to the term $W(s)N(s)X(s)$ in which $N(s)X(s)$ may be discontinuous on the imaginary axis. However, since we assume that D_i is a finite Blaschke product we are able to show below that $\Pi_{\mathbf{K}}(G^*G + F^*F)|_{\mathbf{K}}$ differs from the compression of a continuous function by a finite rank operator, and therefore the essential spectrum is the same as that of the compression of the continuous function. Alternatively, one can say that $\mathcal{T}$ differs from an operator to which we can directly apply the [50] computations by a finite rank operator, so the essential spectra are the same.

This is essentially the same idea used in [50] (as mentioned above) to show that the essential spectra of (the equivalent there of our) $\mathcal{T}$ and $\Pi_{\mathbf{K}}(G^*G + F^*F)|_{\mathbf{K}}$ are the same. This, in turn, is the same idea which was used in [9, p. 16] and [51] for the pure sensitivity problem.

The second complication comes in computing the eigenvalues of $\mathcal{T}$, because G can be an irrational function. This we resolve by means of some additional computations as explained below.

In [50] the idea is to note that $\mathcal{T}$ differs from the multiplication operator $(G^*G + F^*F)|_{\mathbf{K}}$ by a finite rank operator. Then the condition that x_λ is an eigenfunction with eigenvalue λ^2 implies that the image of x_λ under the multiplication operator $\lambda^2 - (G^*G + F^*F)$ lies in a certain finite dimensional subspace of $\mathbf{L}^2$. Computing residues of the image of x_λ under the action of the finite rank operator at certain points in the complex plane allows one to find necessary conditions for λ^2 to be an eigenvalue. This computational procedure was also used in [51] in the pure sensitivity case. Here we follow the same approach, with differing details to account for both more general assumptions and the particulars of the mixed sensitivity problem.

4.4.1 Essential Spectrum of $\mathcal{T}$

By definition,

$$\mathcal{T} = [\Pi_+(A^* + B^*)\Pi_-(A + B) + \Pi_+F^*F] : \mathbf{H}^2 \to \mathbf{H}^2\ , \tag{4.14}$$

where

$$A \triangleq W_1^*W_1(W^*)^{-1}N_i^*D_i^*\ , \quad B \triangleq -WN_oXD_i^*\ . \tag{4.15}$$

Note that by Assumption C, $W_2R^{-1}N_0X \in \mathbf{H}^\infty$.

Now we expand the first half of the right-hand side of (4.14) to obtain

$$\Pi_+(A^* + B^*)\Pi_-(A + B) = \Pi_+A^*\Pi_-A + \Pi_+A^*\Pi_-B + \Pi_+B^*\Pi_-(A + B).$$

Recalling Assumption B.2 that D_i is a finite Blaschke product, we see (see Appendix A) that

$$\Pi_- B : \mathbf{H}^2 \rightarrow \mathbf{H}^2_- \text{ and } \Pi_+ B^* : \mathbf{H}^2_- \rightarrow \mathbf{H}^2$$

are finite rank operators, so

$$\Pi_+ A^* \Pi_- B : \mathbf{H}^2 \rightarrow \mathbf{H}^2$$

and

$$\Pi_+ B^* \Pi_-(A+B) : \mathbf{H}^2 \rightarrow \mathbf{H}^2$$

are finite rank operators. This means that $\mathcal{T}$ is a finite rank perturbation of the operator

$$\mathcal{T}_o = \Pi_+ A^* \Pi_- A + \Pi_+ F^* F.$$

Upon observing this property of $\mathcal{T}$, we have

Theorem 3 *The essential spectrum of $\mathcal{T}$ is*

$$\sigma_e(\mathcal{T}) = \{| W(j\omega) |^2 : j\omega \in \sigma_e(N_i D_i)\} \cup \{\inf_\omega | F(j\omega) |^2, \sup_\omega | F(j\omega) |^2\}.$$

Proof: Since $\mathcal{T}$ is a finite rank perturbation of $\mathcal{T}_0$, we have

$$\sigma_e(\mathcal{T}) = \sigma_e(\mathcal{T}_o).$$

Furthermore,

$$\begin{aligned} A^*A + F^*F &= W_1^* W_1 (W_2^*)^{-1} R^* (W_1^* W_1 (W_2^*)^{-1} R^*)^* + W_1 R (W_1 R)^* \\ &= W_1^* W_1. \end{aligned} \tag{4.16}$$

$W_1 \in \mathbf{H}^\infty$ is continuous on the imaginary axis since it is rational, so the conclusion follows from (the proof of) Theorem 3 in [50]. ∎

Remark 20 *A priori, we might wish to assume that $A^*A + F^*F \in \mathbf{C}$ in order to apply the essential spectral mapping theorem [33, p. 125], as in [50]. However, because of (4.16), we only need the continuity of W_1.*

4.4.2 Eigenvalues of $\mathcal{T}$

Now, in parallel to the developments in [50], we show how to compute the eigenvalues of $\mathcal{T}$. The idea in [50] translated to the present context consists of the following steps:

1. If μ^2 is an eigenvalue of $\mathcal{T}$, then the eigenspace $\mathbf{K}_\mu$ of $\mathcal{T}$ associated with the eigenvalue μ^2 satisfies $\mathbf{K}_\mu \subset (\Phi \mathbf{H}^2)^\perp$, where Φ is an inner function to be defined below.

2. $\mathcal{T} |_{\mathbf{K}_\mu} = W_1^* W_1 + \Delta_\mu$ where Δ_μ is a finite rank operator.

3. The condition for μ^2 to be an eigenvalue is $\mu^2 x_\mu = \mathcal{T} x_\mu$, or

$$(\mu^2 - W_1^* W_1) x_\mu = \Delta_\mu x_\mu. \tag{4.17}$$

4. Since Δ_μ has finite rank, we can find a finite basis for the range of Δ_μ, which allows us to write a matrix representation of Δ_μ. By identifying finitely many μ-dependent points in $\mathbf{C}$ at which we know $\Delta_\mu x_\mu$, we can find necessary and sufficient conditions for μ to be an eigenvalue.

5. Since we are looking only for eigenvalues which may correspond to the norm of $\mathcal{T}$, we need only search for the largest eigenvalue on the interval $(\rho_e(\mathcal{T}), \left\| \begin{pmatrix} G \\ F \end{pmatrix} \right\|_\infty)$.

The following lemma is a modification of [50, Lemma 1]; the proof is basically the same. The only difference is that here we use the finite Blaschke product B_W to "absorb" the unstable poles of $W_1^*(W_2^*)^{-1}R^*$.

Remark 21 *By Assumption C.3, W_1, W_2 and (therefore) R are rational. The existence of B_W and B_μ in the following lemma depend upon this.*

Lemma 4 *Let B_W be the finite Blaschke product which satisfies $B_W W_1^*(W_2^*)^{-1}R^* \in \mathrm{H}^\infty$. Also, let B_μ be the Blaschke product whose zeros are those zeros of $(\mu^2 - F^*F)$ lying in $Re(s) > 0$.*

If μ^2 is an eigenvalue of $\mathcal{T}$, and x_μ a corresponding eigenfunction, then

$$x_\mu \in (B_\mu B_W M H_-^2) \cap H^2,$$

.

Proof: Since $\mu^2 x_\mu = \mathcal{T} x_\mu$, we have

$$(\mu^2 - F^*F)x_\mu = (\Pi_+ M G^* \Pi_- M^* G - \Pi_- F^* F) x_\mu.$$

Multiplying the above equation by $(\mu^2 - F^*F)^{-1} B_\mu^* B_W^* M^*$ we get

$$B_\mu^* B_W^* M^* x_\mu = (\mu^2 - F^*F)^{-1} B_\mu^* B_W^* M^* (\Pi_+ M G^* \Pi_- M^* G - \Pi_- F^* F) x_\mu. \tag{4.18}$$

For any $h \in H^2$,

$$\begin{aligned}
&< B_W^* M^* (\Pi_+ M G^* \Pi_- M^* G - \Pi_- F^* F) x_\mu, h > \\
&\qquad = \; < \Pi_+ M G^* \Pi_- M^* G x_\mu, B_W M h > \\
&\qquad = \; < M G^* \Pi_- M^* G x_\mu, B_W M h > \\
&\qquad = \; < \Pi_- M^* G x_\mu, M^* G B_W M h > \\
&\qquad = \; < \Pi_- M^* G x_\mu, W_1 W_1^* (W_2^*)^{-1} R^* B_W h - B_W W N_o X N_i h > \\
&\qquad = \; 0\,.
\end{aligned}$$

So

$$B_W^* M^* (\Pi_+ M G^* \Pi_- M^* G - \Pi_- F^* F) x_\mu \in \mathbf{H}^2_-.$$

Note also that if z is a zero of $(\mu^2 - F^* F)$, so is $-\bar{z}$. By the construction of B_μ, we see that $(\mu^2 - F^* F)^{-1} B_\mu^*$ is analytic on the left half plane. Therefore

$$(\mu^2 - F^* F)^{-1} B_\mu^* B_W^* M^* (\Pi_+ M G^* \Pi_- M^* G - \Pi_- F^* F) x_\mu \in \mathbf{H}^2_- \ ,$$

and combining this with (4.18) we get

$$B_\mu^* B_W^* M^* x_\mu \in \mathbf{H}^2_-.$$

Therefore

$$x_\mu \in (B_\mu B_W M \mathbf{H}^2_-) \cap \mathbf{H}^2 = (B_\mu B_W M \mathbf{H}^2_-)^\perp \ .$$

∎

The next step is to show that $\mathcal{T}$ differs from the multiplication operator $W_1^* W_1$ by a finite rank operator when restricted to the subspace $\mathbf{K}_\mu = (B_\mu B_W M \mathbf{H}^2)^\perp$. Continuing the parallel to [50], we present this as:

Lemma 5 *$\mathcal{T}|_{\mathbf{K}_\mu}$ is a finite rank perturbation of the multiplication operator $W_1^* W_1|_{\mathbf{K}_\mu}$:*

$$\mathcal{T}|_{\mathbf{K}_\mu} = W_1^* W_1|_{\mathbf{K}_\mu} - \Delta_\mu$$

where

$$\Delta_\mu \triangleq -\Pi_-(A^* A + F^* F) - A^* \Pi_+ A + \Pi_- A^* \Pi_+ A + \Pi_+ A^* \Pi_- B + \Pi_+ B^* \Pi_-(A + B) \quad (4.19)$$

and $\mathrm{rank}(\Delta_\mu) = 2[\mathrm{order}(W_1) + \mathrm{order}(D_i)] + \mathrm{order}(B_\mu)$ *(see Appendix B).*

Proof: We first expand $\mathcal{T}$:

$$\begin{aligned} \mathcal{T} &= [\Pi_+(A^* + B^*)\Pi_-(A + B) + \Pi_+ F^* F] \,|_{H^2} \\ &= \Pi_+ A^* \Pi_- A + \Pi_+ A^* \Pi_- B + \Pi_+ B^* \Pi_-(A + B) + \Pi_+ F^* F. \end{aligned} \quad (4.20)$$

Now we note

$$\begin{aligned} \Pi_+ A^* \Pi_- A &= (A^* - \Pi_- A^*)(A - \Pi_+ A) \\ &= A^* A - A^* \Pi_+ A - \Pi_- A^* A + \Pi_- A^* \Pi_+ A, \\ \Pi_+ F^* F &= F^* F - \Pi_- F^* F \end{aligned} \quad (4.21)$$

Using Lemma 4 we see that $\Pi_+ A$ has finite rank on $\mathbf{K}_\mu$, and we saw in the discussion before Theorem 3 that $\Pi_- B$ and $\Pi_+ B^*$ have finite rank on $\mathbf{H}^2$ and $\mathbf{H}^2_-$, respectively. Therefore we conclude that Δ_μ has finite rank on $\mathbf{K}_\mu$.

Combining (4.20) with (4.19), (4.16) and (4.21), we can write

$$\mathcal{T} = W_1^* W_1 + \Delta_\mu. \quad (4.22)$$

To compute the rank of Δ_μ, we find a basis for the range of Δ_μ: see Appendix B. ∎

4.4.3 Computational Details

In this section, we shall give further details for computing the optimal performance μ, following the ideas of [50]. We take into account the explicit formulas above as well as the complications mentioned before.

Applying (4.22) to an eigenvector x_μ, we can write

$$\begin{aligned}(\mu^2 - W_1^* W_1)x_\mu &= [\Pi_- A^*\Pi_+ A - A^*\Pi_+ A - \Pi_- W_1^* W_1 \\ &\quad +(A^* - \Pi_- A^*)\Pi_- B + \Pi_+ B^*\Pi_- A + \Pi_+ B^*\Pi_- B]x_\mu \qquad (4.23)\end{aligned}$$

In order to use the condition (4.17) to find eigenvalues, we express Δ_μ in terms of a basis for its range when restricted to an eigenspace of $\mathcal{T}$. From Lemma (4) we know that $x_\mu \in (B_\mu B_W M\mathbf{H}^2)^\perp$, so we need only examine the image of $\mathbf{K}_\mu$ under Δ_μ. A direct application of the technique in [50] leads to expressions for a basis in terms of rational functions and M.

The calculations here are an extension of those in [50], so [50] is covered as a special case. Note that in the stable plant case which [50] implicitly treats, $B = 0$ and G is rational.

In [50], the authors state that $\mathrm{rank}(\Delta_\mu) \leq 2N$, where (translating to our notation) $N = \mathrm{order}(G) + \mathrm{order}(F)$. They go on to treat in their calculations the "generic" case in which the poles and zeros of G, G^*, F, F^* and M are all distinct, stating that in this case $\mathrm{rank}(\Delta_\mu) = 2N$. The simplification (4.16) shows that for the mixed sensitivity problem the "non-generic" case $\mathrm{rank}(\Delta_\mu) < 2N$ is in fact generic! This observation shows that the equation (18) in [50] gives insufficiently many equations to solve the problem. Part of the solution is to reduce the required number of basis elements, and this is the point of Proposition 4 below, which shows that the Laurent series coefficients of x_μ at each pole of B_W are zero.

In the following calculation, we shall use $R_n^f(p)$ to denote the $(-n)^{th}$ coefficient in the Laurent expansion of the meromorphic function f at its pole p. We shall also use $R^f(p)$ to denote the residue ($(-1)^{st}$ coefficient in the Laurent expansion) of f at p. Also, $\mathcal{P}(f)$ and $\mathcal{Z}(f)$ mean, respectively, the poles and zeros of f. (Since we assume f to be meromorphic here, there is no concern about non-ordinary zeros such as may occur on the boundary of a region of analyticity.)

Remark 22 *Since $W(s)$ is outer by construction, we have $\mathcal{Z}(W^*) \subset$* (right half plane).

Lemma 6 *The poles of $\frac{W_1}{W}$ are in the left half plane and $\mathcal{P}(\frac{W_1}{W}) = \mathcal{P}(\frac{W_1^*}{W^*}B_W) = \mathcal{P}(B_W)$.*

Proof: We know that $\mathcal{P}(\frac{W_1^*}{W^*}) \subset \mathcal{Z}(W^*) \cup \mathcal{P}(W_1^*)$. If $p \in \mathcal{P}(W_1^*)$, since $W_1 \in H^\infty$, we have $p \in$(right half plane). Similarly, $\mathcal{Z}(W^*) \subset$ (right half plane), since $W(s)$ is outer by construction. Furthermore, since $W^*W = W_1^*W_1 + W_2^*W_2$, if $W_1^*W_1(j\omega) > 0$, we have $W^*W(j\omega) > 0$, and so $W(s)$ has no zeros on the imaginary axis. So $\mathcal{P}(\frac{W_1^*}{W^*}) \subset$ (right half plane). Noting that B_W is the Blaschke product such that $\frac{W_1^*}{W^*}B_W \in \mathbf{H}^\infty$, and using the notation $-\overline{\mathcal{P}} = \{p : -\overline{p} \in \mathcal{P}\}$, we see that the poles of $\frac{W_1^*}{W^*}B_W$ are in the left half plane and $\mathcal{P}(\frac{W_1^*}{W^*}B_W) = -\overline{\mathcal{P}(\frac{W_1^*}{W^*})}$. Therefore $\mathcal{P}(\frac{W_1}{W}) = -\overline{\mathcal{P}(\frac{W_1^*}{W^*})} = \mathcal{P}(\frac{W_1^*}{W^*}B_W) = \mathcal{P}(B_W)$. ∎

Remark 23 *We note that* $(W_1/W_2)^* R^* = (W_1/W)^*$ *and that* W_1/W *is an outer spectral factor of* $\frac{W_1 W_1^*}{W_1 W_1^* + W_2^* W_2}$. *The poles of* W_1/W *are the zeros of* W, *and therefore the right-half-plane poles of* $\frac{W_1 W_1^*}{W_1 W_1^* + W_2^* W_2}$ *are the zeros of* W^*, *which are the zeros of* B_W. *Thus the poles of* B_W *are the poles of* R. *If* W_1 *and* W_2 *have no common zeros,* $\text{zeros}(W) \cap \text{zeros}(W_1) = \emptyset$. *In general,* $\text{poles}(B_W) = \text{zeros}(W) \setminus (\text{zeros}(W_1) \cap \text{zeros}(W_2))$.

Proposition 4 *For any pole* $\eta \in \mathcal{P}(B_W)$, *if* $\eta \notin \mathcal{P}(D_i) \cup \mathcal{P}(N_i)$, *then*

$$R_m^{x_\mu}(\eta) = 0.$$

for all $m \geq 1$.

Proof: From (4.23) we can write

$$(\mu^2 - W_1^* W_1)x_\mu = -A^* \Pi_+ A x_\mu + A^* b_- + \Pi_+ B^* c_- + a_-,$$

where

$$\begin{aligned} a_- &= \Pi_-(-W_1^* W_1 - A^* \Pi_- B + A^* \Pi_+ A) x_\mu \\ b_- &= \Pi_- B x_\mu \\ c_- &= \Pi_-(A + B) x_\mu. \end{aligned}$$

From Lemma 5 and the definition of A, if η is an n^{th} order pole of B_W, it also is a pole of A^* of order $\ell \geq n$. Then

$$\lim_{s \to \eta} (s - \eta)^\ell A^* h(s) = R_\ell^{A^*}(\eta) h(\eta) \qquad \forall h \in H_-^2. \tag{4.24}$$

Furthermore,

$$\lim_{s \to \eta} (s - \eta)^{\ell+1} A^* b_- = \lim_{s \to \eta} (s - \eta) R_\ell^{A^*}(\eta) b_- = 0$$

since b_- is analytic in the left half plane. We have

$$\lim_{s \to \eta} (s - \eta)^{\ell - n + 1} \Pi_+ B^* c_- = 0$$

since $\Pi_+ B^* c_-$ can have poles only at the poles of D_i. Now

$$\lim_{s \to \eta} (s - \eta)^{n+1} (\mu - W_1^* W_1) x_\mu = 0$$

since W_1 and B_W have no poles in common, and therefore

$$\lim_{s \to \eta} (s - \eta)^{\ell + m} A^* \Pi_+ A x_\mu = 0 \tag{4.25}$$

for every $m \geq 1$. Since η is an n^{th}-order pole of B_W, and B_W has no poles in common with A, we have

$$\Pi_+ A x_\mu = \frac{W_1^* W_1}{W^*}(\eta) N_i^*(\eta) D_i^*(\eta) \sum_{m=1}^{n} \frac{R_m^{x_\mu}(\eta)}{(s - \eta)^m} + a(s), \tag{4.26}$$

where $a(s)$ is analytic at $s = \eta$. So from (4.24), (4.25) and (4.26) (taking $m = n$ in (4.25)) we have

$$R_\ell^{A^*}(\eta)\frac{W_1^*W_1}{W^*}(\eta)N_i^*(\eta)D_i^*(\eta)R_n^{x_\mu}(\eta) = 0.$$

Therefore $R_n^{x_\mu}(\eta) = 0$. But then

$$\Pi_+ Ax_\mu = \frac{W_1^*W_1}{W^*}(\eta)N_i^*(\eta)D_i^*(\eta)\sum_{m=1}^{n-1}\frac{R_m^{x_\mu}(\eta)}{(s-\eta)^m} + a(s),$$

and $R_{n-1}^{x_\mu}(\eta) = 0$. Similarly we see $R_m^{x_\mu}(\eta) = 0$ for each $m \geq 1$. ∎

By Lemma 4, $x_\mu = B_\mu B_W N_i D_i h_-$ for some $h_- \in \mathbf{H}^2_-$. Using this, in Appendix B we explicitly compute the terms in the right side of the expansion (4.23) for Δ_μ.

4.5 Example: The Damped Flexible Beam

We next illustrate the technique developed in the previous sections using the transfer function model of a damped flexible beam presented in Section 2.1.

For simplicity, we shall use the following sensitivity and complementary sensitivity weighting functions, $W_1 = \frac{1}{a+s}, W_2 = \epsilon(b+s)$, where $a, b > 0$. We note that these weighting functions emphasize low and high frequency, respectively, as good design practice would dictate (see, *e.g.*, [27], [40]).

Taking ξ_1^2 and ξ_2^2 to be the zeros of

$$\begin{aligned} W^*W &\triangleq W_1^*W_1 + W_2^*W_2 \\ &= \frac{1}{a^2 - s^2} + \epsilon^2(b^2 - s^2) \end{aligned}$$

we can write

$$W^*W = \frac{\epsilon^2(\xi_1 - s)(\xi_2 - s)(\xi_1 + s)(\xi_2 + s)}{(a-s)(a+s)}$$

We note that $\frac{1}{\epsilon^2} + (b^2 - s^2)(a^2 - s^2)$ has real coefficients. Since it cannot have pure imaginary roots, we see that it has exactly two roots with positive real parts. We can assume that $Re\xi_1 > 0, Re\xi_2 > 0$. If ξ_1 is complex, then $\xi_2 = \bar{\xi}_1$.

Thus for this example we can explicitly write

$$\begin{aligned} W &= \frac{\epsilon(\xi_1 + s)(\xi_2 + s)}{a + s}, \\ \frac{W_1^*}{W^*} &= \frac{1}{\epsilon(\xi_1 - s)(\xi_2 - s)}, \\ B_W &= \frac{(\xi_1 - s)(\xi_2 - s)}{(\xi_1 + s)(\xi_2 + s)}, \end{aligned}$$

$$
\begin{aligned}
F^*F &= \frac{b^2 - s^2}{(\xi_1 - s)(\xi_2 - s)(\xi_1 + s)(\xi_2 + s)} \\
F &= \frac{b + s}{(s + \xi_1)(s + \xi_2)} \\
G &= \frac{1}{\epsilon(s + a)(s - \xi_1)(s - \xi_2)} - \frac{\epsilon(s + \xi_1)(s + \xi_2)}{(s + a)} \cdot \frac{N(s)}{N(0)} \\
M &= N_i(s) \\
A &= \frac{1}{\epsilon(s + a)(s - \xi_1)(s - \xi_2)} \\
B &= -\frac{\epsilon(s + \xi_1)(s + \xi_2)}{(s + a)} \cdot \frac{N(s)}{N(0)}
\end{aligned}
$$

4.5.1 Essential Spectrum

$N_i D_i$ has no essential singularity on $j\mathbf{R}$, so by Theorem 1, we find that the essential spectral radius is given by

$$
\begin{aligned}
\rho_e(\mathcal{T}) &= \sup_{\omega} |F(j\omega)|^2 \\
&= \begin{cases} \frac{\epsilon^2 b^2}{1+\epsilon^2 a^2 b^2}, & \text{if} \quad \frac{1}{\epsilon} - b^2 \le 0 \\ \frac{\epsilon}{2+\epsilon(a^2 - b^2)}, & \text{if} \quad \frac{1}{\epsilon} - b^2 > 0 \end{cases}
\end{aligned}
$$

following a simple calculation to determine the zeros of $\frac{d}{d\omega} \mid F(j\omega) \mid^2$.

4.5.2 Eigenvalues

Assume $\mu^2 > \sigma_e(\mathcal{T}) = \sup_\omega |F(j\omega)|^2$, so $\mu^2 - F^*F(j\omega) > 0$. This means that $\mu^2 - F^*F = 0$ does not have pure imaginary solutions, and so it has exactly 2 roots with positive real parts which we shall denote as s_1 and s_2.

$$
B_\mu = \frac{(s - s_1)(s - s_2)}{(s + s_2)(s + s_1)},
$$

where we can write

$$
\begin{aligned}
\mu^2 \epsilon^2 (s - s_1)(s - s_2)(s + s_1)(s + s_2) &= \mu^2[1 + \epsilon^2(b^2 - s^2)(a^2 - s^2)] - \epsilon^2(b^2 - s^2) \\
&= \mu^2 \epsilon^2 (\xi_1 - s)(\xi_2 - s)(\xi_1 + s)(\xi_2 + s) - \epsilon^2(b^2 - s^2).
\end{aligned}
$$

In this case, for any $h \in \mathbf{H}^2$ and any $h_- \in \mathbf{H}^2_-$, $\Pi_- B h = 0$ and $\Pi_+ B^* h_- = 0$, so (4.23) becomes

$$
(\mu^2 - W_1^* W_1) x_\mu = (\Pi_- A^* \Pi_+ A - A^* \Pi_+ A - \Pi_- W_1^* W_1) x_\mu, \tag{4.27}
$$

where $A = \frac{W_1^* W_1}{W^*} N_i^* = \frac{N_i^*}{\epsilon(\xi_1 - s)(\xi_2 - s)(a + s)}$.

From Appendix B, we see that $k = 1, m = 2$ and $n = 0$, and the corresponding $\phi_j, j = 1,2,3,4$ are

$$\frac{1}{s-a}, \frac{A^*}{s+a}, \frac{A^*}{s+s_1}, \frac{A^*}{s+s_2}.$$

So we can write down the matrix $\mathcal{A}(\mu)$ easily as

$$\mathcal{A}(\mu) = \begin{pmatrix} \frac{1}{\zeta-a} & \frac{A^*(\zeta)}{\zeta+a} & \frac{A^*(\zeta)}{\zeta+s_1} & \frac{A^*(\zeta)}{\zeta+s_2} \\ \frac{1}{-\zeta-a} & \frac{A^*(-\zeta)}{-\zeta+a} & \frac{A^*(-\zeta)}{-\zeta+s_1} & \frac{A^*(-\zeta)}{-\zeta+s_2} \\ 0 & \frac{1}{a-\xi_1} & -\frac{1}{-s_1+\xi_1} & -\frac{1}{-s_2+\xi_1} \\ 0 & \frac{1}{a-\xi_2} & -\frac{1}{-s_1+\xi_2} & -\frac{1}{-s_2+\xi_2} \end{pmatrix}.$$

Then $\det[\mathcal{A}(\mu)] = 0$ is equivalent to

$$\frac{N_i(-\zeta)}{(\xi_1-\zeta)(\xi_2-\zeta)} \det \begin{pmatrix} \frac{1}{a-\zeta} & \frac{1}{s_1-\zeta} & \frac{1}{s_2-\zeta} \\ \frac{1}{a-\xi_1} & \frac{1}{s_1-\xi_1} & \frac{1}{s_2-\xi_1} \\ \frac{1}{a-\xi_2} & \frac{1}{s_1-\xi_2} & \frac{1}{s_2-\xi_2} \end{pmatrix} = \frac{N_i(\zeta)}{(\xi_1+\zeta)(\xi_2+\zeta)} \det \begin{pmatrix} \frac{1}{a+\zeta} & \frac{1}{s_1+\zeta} & \frac{1}{s_2+\zeta} \\ \frac{1}{a-\xi_1} & \frac{1}{s_1-\xi_1} & \frac{1}{s_2-\xi_1} \\ \frac{1}{a-\xi_2} & \frac{1}{s_1-\xi_2} & \frac{1}{s_2-\xi_2} \end{pmatrix} \tag{4.28}$$

We solve the above equation for the largest $\mu^2 \in (\sigma_e(\mathcal{T}), \frac{1}{a^2})$.

Remark 24 *In the stable plant case, if $\mu^2 = \frac{1}{a^2}$, then $\zeta = 0$, so $\mu^2 = \frac{1}{a^2}$ is a solution to (4.28). The open loop case (i.e., with compensator $C = 0$) corresponds to this solution.*

Remark 25 *In the present case, $D_i = 1$. Since the optimal performance of the weighted mixed sensitivity doesn't depend on the outer part of the denominator of the plant, so we see that (4.28) doesn't depend on the instability of the plant at all! In other words, with a stable plant $P_s = N_i$, the corresponding equation (4.28) will be identical with that of the present case, so the optimal performances for both cases will be the same. We know that for the stable plant case the optimal performance $\mu_s \in (\sigma_e(\mathcal{T}), \frac{1}{a^2})$, so the optimal performance for the present unstable case $\mu = \mu_s \in (\sigma_e(\mathcal{T}), \frac{1}{a^2})$.*

Remark 26 *If $N_i^*(-a) = 0$, repeating the above procedure we have*

$$\mathcal{B}_1 \begin{bmatrix} R^{x\mu}(-s_1) \\ R^{x\mu}(-s_2) \end{bmatrix} = 0,$$

where

$$\mathcal{B}_1 \triangleq \begin{bmatrix} \frac{1}{s_1-\xi} & \frac{1}{s_2-\xi} \\ \frac{1}{s_1-\xi_2} & \frac{1}{s_2-\xi_2} \\ \frac{f_b(\zeta_1)}{s_1+\zeta_1} - \frac{f_b(\zeta_2)}{s_1+\zeta_2} & \frac{f_b(\zeta_1)}{s_2+\zeta_1} - \frac{f_b(\zeta_2)}{s_2+\zeta_2} \end{bmatrix}$$

In order to have nontrivial solution, we require that

$$\operatorname{rank}(\mathcal{B}_1) = 1.$$

First

$$\det \begin{bmatrix} \frac{1}{s_1-\xi} & \frac{1}{s_2-\xi} \\ \frac{1}{s_1-\xi_2} & \frac{1}{s_2-\xi_2} \end{bmatrix} = 0 \Longleftrightarrow (\xi_2-\xi_1)(s_1-s_2) = 0$$

So either $\xi_2 = \xi_1$ *or* $s_1 = s_2$.
It is easy to see that

$$\xi_2 = \xi_1 \Longleftrightarrow (a^2 - b^2)^2 = \frac{4}{\epsilon^2}$$

$$s_1 = s_2 \Longleftrightarrow (b^2 - a^2 + \frac{1}{\mu\epsilon^2})^2 = \frac{4}{\epsilon^2}.$$

If $s_1 = s_2$ *we can see that in fact in this case* $\mathrm{rank}(B_1) = 1$, *so we can solve*

$$(b^2 - a^2 + \frac{1}{\mu\epsilon^2})^2 = \frac{4}{\epsilon^2}.$$

for μ.
If $\xi_2 = \xi$, *then* $\mathrm{rank}(B_1) = 1$, *if and only if*

$$\frac{N_i^*(\zeta_1)}{(\xi_1+\zeta_1)^2} \det \begin{bmatrix} \frac{1}{s_1-\xi} & \frac{1}{s_2-\xi} \\ \frac{1}{s_1+\zeta_1} & \frac{1}{s_2+\zeta_1} \end{bmatrix} = \frac{N_i^*(\zeta_2)}{(\xi_1+\zeta_2)^2} \det \begin{bmatrix} \frac{1}{s_1-\xi} & \frac{1}{s_2-\xi} \\ \frac{1}{s_1+\zeta_2} & \frac{1}{s_2+\zeta_2} \end{bmatrix}.$$

4.6 Conclusion

We have shown that the results and techniques of [50] can be modified to solve the $\mathbf{H}^\infty$ optimal mixed sensitivity problem for a very general SISO unstable plants. Part of the extension is needed to treat the model of a damped flexible beam [11], which has irrational outer part. It is also the case that when N_i is rational and D_i general inner, we can solve the problem by the same techniques given in the previous sections. We have observed that the complementary sensitivity weighting function should be taken to be as improper as the plant is strictly proper. Because of this, an important area of continuing work is the case of irrational complementary sensitivity weighting function.

Chapter 5

MIMO Mixed Sensitivity

5.1 Introduction

The problem of computing the optimal performance for infinite dimensional MIMO systems is different from that of the SISO case mainly because we no longer enjoy the commutativity of scalars. Under some commutation conditions, [35] studied the computation of the optimal performance of a class of infinite dimensional MIMO systems.

In this chapter, we consider $\mathbf{H}^\infty$-optimal mixed sensitivity design for an idle speed control model posed in [46]. This is a three-input four-output system, and the infinite dimensional parts of the system are time delays with different delay parameters. In [46], two first order lags were used to approximate these two delays. Following now-standard transformation of the problem to an operator norm problem, in order to solve this design problem without using rational approximation of any kind, first we have to be able to compute the inner-outer factorization explicitly for an irrational $\mathbf{H}^\infty$ matrix. Also in this case, the commutation conditions of [35] can not be satisfied due to the two different time delays, so that the the techniques and results developed there cannot be applied to this design problem. We resolve this difficulty by a more detailed characterization of the eigenspace of a certain operator. This characterization enables us to compute the eigenvalues and eigenfunctions of the operator explicitly. Also, by using a result proved in [50], we decompose the operator on three orthogonal subspaces of $\mathbf{H}^2$ and compute the essential spectra explicitly.

Although in this chapter we restrict ourselves to $\mathbf{H}^\infty$-optimal mixed sensitivity design of this particular model, the method itself applies more generally.

5.2 Problem Description

In Section 2.4 we presented a model of idle speed control for a fuel injected engine. This model involved a system having two incommensurate delays, and multiple inputs and outputs.

Our goal is to design a feedback compensator controlling the valve input and spark advance to minimize the effect of the torque load disturbance, and at the same time guarantee a good stability margin. First we can transform the problem into the standard compensation configuration shown if Figure 5.1. d is a disturbance at the plant output, u is the control

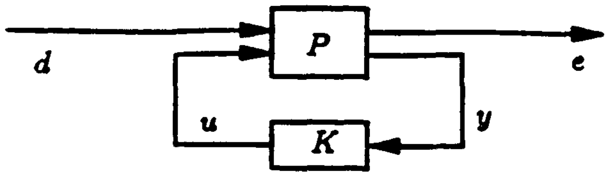

Figure 5.1: Standard Compensation Configuration

input, which consists of two components: idle valve setting and ignition timing setting, y is the speed output, e combines the three signals (the two control inputs u and the speed output y) to be constrained.

We want to design a compensator $K = (k_1, k_2)^T$ for the closed-loop shown in Figure 5.1 so that $\| \begin{pmatrix} w_1 f_{e_1 d} \\ w_2 f_{e_2 d} \\ w_3 f_{e_3 d} \end{pmatrix} \|_\infty$ achieves a minimum under the constraint of the closed-loop internal stability. We shall denote this minimum by μ_o and call it the optimal performance. w_1, w_2 and w_3 are weighting functions with w_1 a high-pass filter and w_2, w_3 low-pass filters. We shall use the following weighting functions as in [46]:

$$w_1 = \frac{c_1 s}{s + r_1}, w_2 = c_2 \frac{s + \theta}{s + r_2}, w_3 = \frac{c_3}{s + r_3}.$$

5.3 Standard Formulation of the Problem

In this section we transform the problem into a problem of computing an operator norm. The basic ideas of the transformation procedure we use here is by now standard: First we parameterize ([5], [48]) all stabilizing compensators and transform the problem into an infimal norm problem of the form:

$$\mu_o = \min_{Z \in \mathbf{H}^\infty_{2\times 1}} \| \begin{pmatrix} X - Z \\ C \end{pmatrix} \|_\infty,$$

where $X \in \mathbf{L}^\infty_{2\times 1}$ and $C \in \mathbf{H}^\infty$. In the course of this transformation, an inner-outer factorization of a $\mathbf{H}^\infty$ matrix is needed. Although the existence of the inner-outer factorization has been proven theoretically [43], explicity writing down the inner-outer factorization is in general difficult even in the scalar case. We observe that for a quite general case[47], we only need an inner-outer factorization of an upper (or lower) triangular or a Hermitian matrix. In the present case, for example, what we need is to explicitly calculate the inner and outer factors of a two-by-two upper triangular $\mathbf{H}^\infty$ matrix. In this section we also show how to get this factorization explicitly.

The next step is to use a well-known result of [36] to further transform this minimization problem into an operator norm problem. Since the disturbance d is a scalar in this case, we shall see that the operator we get is a scalar operator on $\mathbf{H}^2$.

5.3.1 A Preliminary Transformation

Since P is stable, using [21, pp.822-824], with a stabilizing compensator K we have

$$f_{ed} = P_{11} - P_{12}QP_{21} = \begin{pmatrix} 0 \\ 0 \\ p_1 \end{pmatrix} - p_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ p_2 & p_3 \end{pmatrix} Z,$$

where $Z \in \mathbf{H}^{\infty}_{2\times 1}$.

So the problem becomes

$$\mu_o = \min_{Z \in \mathbf{H}^{\infty}_{2\times 1}} \| \begin{pmatrix} 0 \\ 0 \\ w_3 p_1 \end{pmatrix} - \begin{pmatrix} w_1 p_1 & 0 \\ 0 & w_2 p_1 \\ w_3 p_1 p_2 & w_3 p_1 p_3 \end{pmatrix} Z \|_{\infty},$$

where w_1 is a high-pass filter, w_2 and w_3 are low-pass filters. μ_o is the optimal performance.

Without loss of generality we can assume $\tau_v > \tau_s$ as in the present case, and let $\tau = \tau_v - \tau_s$. Then

$$\begin{aligned} \mu_o &= \min_{Z \in \mathbf{H}^{\infty}_{2\times 1}} \| \begin{pmatrix} 0 \\ 0 \\ w_3 p_1 \end{pmatrix} - \begin{pmatrix} w_1 p_1 & 0 \\ 0 & w_2 p_1 \\ w_3 p_1 p_2 & w_3 p_1 p_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Z \|_{\infty} \\ &= \min_{Z \in \mathbf{H}^{\infty}_{2\times 1}} \| \begin{pmatrix} 0 \\ 0 \\ w_3 p_1 \end{pmatrix} - \begin{pmatrix} 0 & w_1 p_1 \\ w_2 p_1 & 0 \\ w_3 p_1 p_3 & w_3 p_1 p_2 \end{pmatrix} Z \|_{\infty}. \end{aligned}$$

Let $T_1 = \begin{pmatrix} 0 \\ 0 \\ w_3 p_1 \end{pmatrix}$ and $T_2 = p_1 \begin{pmatrix} 0 & w_1 \\ w_2 & 0 \\ w_3 p_3 & w_3 p_2 \end{pmatrix}$. We wish to find $A = (a_{ij})_{2\times 1}$ and $B = (b_{ij})_{2\times 2}$ satisfying

$$\left\{ \begin{array}{l} A^* B = T_1^* T_2, \\ B^{(*)} = T_2^{(*)}. \end{array} \right. \tag{5.1}$$

Solving (5.1) we get

$$\left\{ \begin{array}{c} b_{11}^{(*)} = p_1^{(*)}(w_2^{(*)} + w_3^{(*)} p_3^{(*)}), \\ b_{12} = \frac{w_3^{(*)} p_1^{(*)} p_2 p_3^*}{b_{11}^*}, \\ b_{21} = 0, \\ b_{22}^{(*)} = p_1^{(*)}(w_1^{(*)} + w_3^{(*)} p_2^{(*)}) - b_{12}^{(*)}. \end{array} \right.$$

$$\left\{ \begin{array}{c} a_1 = \frac{w_3^{(*)} p_1^{(*)} p_3^*}{b_{11}^*}, \\ a_2 = \frac{w_3^{(*)} p_1^{(*)} p_2^* - a_1 b_{12}^*}{b_{22}^*}. \end{array} \right.$$

$$b_{11} = \frac{\beta_s \kappa_t (s + \theta_s)(e_1 - s) \cdots (e_4 - s)}{(\pi_{s1} + s)^2 (\pi_{s2} + s)^2 (r_2 + s)(r_3 + s)},$$

$$b_{12} = \frac{c_3^2 \beta_s^2 \beta_v \kappa_s \kappa_t (\theta_s^2 - s^2)(r_2 - s)}{(r_3 + s)(\pi_{s1} + s)(\pi_{s2} + s)(\pi_{v1} + s)(\pi_{v2} + s)(e_1 + s) \cdots (e_4 + s)} e^{-s\tau},$$

$$b_{21} = 0,$$

$$b_{22} = \frac{(\theta_s + s)(h_1 + s)\cdots(h_8 + s)}{(\pi_{s1} + s)(\pi_{s2} + s)(\pi_{v1} + s)(\pi_{v2} + s)(r_1 + s)(r_3 + s)(e_1 + s)\cdots(e_4 + s)},$$

where $e_1, ..., e_4$ are the four roots of

$$c_2^2(\theta^2 - s^2)(r_3^2 - s^2)(\pi_{s1}^2 - s^2)(\pi_{s2}^2 - s^2) + c_3^2\beta_s^2\kappa_s^2(\theta_s^2 - s^2)(r_2^2 - s^2) = 0$$

with positive real parts. and $h_1, ..., h_8$ are the eight roots of

$$[c_3^2\beta_v^2(r_1^2 - s^2) - c_1^2 s^2(r_3^2 - s^2)(\pi_{v1}^2 - s^2)(\pi_{v2}^2 - s^2)](e_1^2 - s^2)\cdots(e_4^2 - s^2) -$$
$$-c_3^4\beta_s^2\beta_v^2\kappa_s^2(\theta_s^2 - s^2)(r_1^2 - s^2)(r_2^2 - s^2) = 0$$

with positive real parts.

Further let $C^{(*)} = T_1^{(*)} - A^{(*)} = w_3^{(*)}p_1^{(*)} - A^{(*)}$.

It is easy to check that $(T_1 - T_2Z)^{(*)} = (A - BZ)^{(*)} + C^{(*)}$. Noting that

$$B = \begin{pmatrix} (b_{11})_i & \frac{b_{12}}{b_{22}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (b_{11})_o & 0 \\ 0 & b_{22} \end{pmatrix}$$

and $\frac{b_{12}}{b_{22}} \in \mathbf{H}^\infty$, we have

$$\mu = \min_{Z \in \mathbf{H}^\infty_{2\times 1}} \| \begin{pmatrix} A - BZ \\ C \end{pmatrix} \|_\infty = \min_{Z \in \mathbf{H}^\infty_{2\times 1}} \| \begin{pmatrix} A - DZ \\ C \end{pmatrix} \|_\infty,$$

where

$$D = (d_{ij}) \triangleq \begin{pmatrix} (b_{11})_i & \frac{b_{12}}{b_{22}} \\ 0 & 1 \end{pmatrix}.$$

5.3.2 Inner-Outer Factorization of D

In the following, we explain how to get the inner-outer factorization of D. The key here is that we observe that each element of the outer factor for this matrix takes the form $Q_1 + Q_2 e^{-s\tau}$, where Q_1 and Q_2 are rational.

Let

$$D = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} =: UV \tag{5.2}$$

be the inner-outer factorization of D. Since $det(D) = d_{11} = det(U)det(V)$, and d_{11} is inner, we know [39] $det(U) = \alpha d_{11}, det(V) = \bar{\alpha}$, where $|\alpha| = 1$. Of course we can set $\alpha = 1$. So from (5.2) and $U^{(*)} = I$ we have

$$U = \begin{pmatrix} d_{11}v_1^* & d_{11}v_3^* \\ -v_3 & v_1 \end{pmatrix}.$$

and

$$v_1^{(*)} + v_3^{(*)} = 1 \tag{5.3}$$
$$d_{11}(v_1^* v_2 + v_3^* v_4) = d_{12} \tag{5.4}$$
$$v_1 v_4 - v_2 v_3 = 1 \tag{5.5}$$

Let $v_i = P_{v_i} + Q_{v_i} e^{-s\tau}, i = 1, \ldots, 4, \tau = \tau_v - \tau_s$. Then from (5.3) we have

$$\begin{aligned} P_{v_1}^{(*)} + Q_{v_1}^{(*)} + P_{v_3}^{(*)} + Q_{v_3}^{(*)} &= 1 && (5.6) \\ P_{v_1}^* Q_{v_1} + P_{v_3}^* Q_{v_3} &= 0 && (5.7) \end{aligned}$$

From (5.4) we have

$$\begin{aligned} P_{v_1}^* P_{v_2} + Q_{v_1}^* Q_{v_2} + P_{v_3}^* P_{v_4} + Q_{v_3}^* Q_{v_4} &= 0 && (5.8) \\ Q_{v_1}^* P_{v_2} + Q_{v_3}^* P_{v_4} &= 0 && (5.9) \\ d_{11}(P_{v_1}^* Q_{v_2} + P_{v_3}^* Q_{v_4}) &= R && (5.10) \end{aligned}$$

where R is defined by $d_{12} = Re^{-s\tau}$. From (5.5) we have

$$\begin{aligned} P_{v_1} Q_{v_4} + P_{v_4} Q_{v_1} = P_{v_2} Q_{v_3} + P_{v_3} Q_{v_2} && (5.11) \\ Q_{v_1} Q_{v_4} = Q_{v_2} Q_{v_3} && (5.12) \\ P_{v_1} P_{v_4} - P_{v_2} P_{v_3} = 1 && (5.13) \end{aligned}$$

Now from (5.7), (5.9) and (5.13), we get $Q_{v_1} = Q_{v_3} = 0$. So (5.6), (5.8) and (5.13) become

$$\begin{cases} P_{v_1}^{(*)} + P_{v_3}^{(*)} = 1, \\ P_{v_1} P_{v_2}^* + P_{v_3} P_{v_4}^* = 0, \\ P_{v_1} P_{v_4} - P_{v_2} P_{v_3} = 1, \end{cases}$$

and we get $P_{v_1} = \frac{P_{v_4}^*}{\Lambda}, P_{v_3} = -\frac{P_{v_2}^*}{\Lambda}$, where $\Lambda = P_{v_2}^{(*)} + P_{v_4}^{(*)}$. But $\frac{1}{\Lambda} = \frac{P_{v_4}^{(*)}}{\Lambda^2} + \frac{P_{v_2}^{(*)}}{\Lambda^2} = P_{v_1}^{(*)} + P_{v_3}^{(*)} =$ 1, so $\Lambda = 1$. Therefore $P_{v_1} = P_{v_4}^*, P_{v_3} = -P_{v_2}^*$. From (5.10) and (5.11) we have

$$\begin{cases} P_{v_1}^* Q_{v_2} + P_{v_3}^* Q_{v_4} = d_{11}^* R, \\ -P_{v_3} Q_{v_2} + P_{v_1} Q_{v_4} = 0. \end{cases}$$

So $Q_{v_2} = d_{11}^* R P_{v_1}, Q_{v_4} = d_{11}^* R P_{v_3}$.

Now the problem becomes to find P_{v_1} and P_{v_3} such that

$$\begin{cases} P_{v_1}^{(*)} + P_{v_3}^{(*)} = 1, \\ P_{v_1}, P_{v_3} \in \mathbf{H}^\infty, \\ v_4 = P_{v_1}^* + d_{11}^* P_{v_3} R e^{-s\tau} \in \mathbf{H}^\infty, \\ v_2 = P_{v_3}^* + d_{11}^* P_{v_1} R e^{-s\tau} \in \mathbf{H}^\infty. \end{cases} \qquad (5.14)$$

If $d_{11} = \frac{(\delta_1 - s)\cdots(\delta_k - s)}{(\delta_1 + s)\cdots(\delta_k + s)}$, we define $P_{v_1}^* = \frac{f^*(s)}{(\delta_1 - s)\cdots(\delta_k - s)}, P_{v_3}^* = \frac{g^*(s)}{(\delta_1 - s)\cdots(\delta_k - s)}$, where

$$f^*(s) = \xi_k s^k + \xi_{k-1} s^{k-1} + \cdots + \xi_0,$$

$$g^*(s) = \zeta_k s^k + \zeta_{k-1} s^{k-1} + \cdots + \zeta_0.$$

In (5.14) we have

$$\begin{cases} v_4 = \frac{f^*(s)}{(\delta_1 - s)\cdots(\delta_k - s)} + \frac{g(s)R(s)e^{-s\tau}}{(\delta_1 - s)\cdots(\delta_k - s)} \in \mathbf{H}^\infty, \\ v_2 = -\frac{g^*(s)}{(\delta_1 - s)\cdots(\delta_k - s)} + \frac{f(s)R(s)e^{-s\tau}}{(\delta_1 - s)\cdots(\delta_k - s)} \in \mathbf{H}^\infty. \end{cases}$$

So we require

$$f^{*}(\delta_i) + R(\delta_i)e^{-\delta_i \tau} g(\delta_i) = 0 \tag{5.15}$$

$$-g^{*}(\delta_i) + R(\delta_i)e^{-\delta_i \tau} f(\delta_i) = 0 \tag{5.16}$$

$i = 1, \ldots, k$. Now we show that $P_{v_1}^{(*)} + P_{v_2}^{(*)} = 1$ is satisfied provided that

$$\phi = |\xi_k|^2 + |\zeta_k|^2 = 1. \tag{5.17}$$

In fact, multiply (5.15) by $f(\delta_i)$, (5.16) by $-g(\delta_i)$, and add them up, we see

$$f^{(*)}(\delta_i) + g^{(*)}(\delta_i) = 0.$$

By the definition of involution $*$, we also see

$$f^{(*)}(-\bar{\delta}_i) + g^{(*)}(-\bar{\delta}_i) = 0.$$

So $f^{(*)}(s) + g^{(*)}(s) = \phi(\delta_1 - s)\cdots(\delta_k - s)(\bar{\delta}_1 + s)\cdots(\bar{\delta}_k + s)$, and therefore

$$P_{v_1}^{(*)} + P_{v_2}^{(*)} = \frac{f^{(*)}(s) + g^{(*)}(s)}{(\delta_1 - s)\cdots(\delta_k - s)(\bar{\delta}_1 + s)\cdots(\bar{\delta}_k + s)} = \phi = 1.$$

In summary, we solve (5.15), (5.16) and (9) to get $f(s)$ and $g(s)$, and

$$V = \begin{pmatrix} \frac{f(s)}{(\bar{\delta}_1+s)\cdots(\bar{\delta}_k+s)} & -\frac{g^{*}(s)-f(s)R(s)e^{-s\tau}}{(\delta_1-s)\cdots(\delta_k-s)} \\ \frac{g(s)}{(\bar{\delta}_1+s)\cdots(\bar{\delta}_k+s)} & \frac{f^{*}(s)+g(s)R(s)e^{-s\tau}}{(\delta_1-s)\cdots(\delta_k-s)} \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{f^{*}(s)}{(\bar{\delta}_1+s)\cdots(\bar{\delta}_k+s)} & \frac{g^{*}(s)}{(\bar{\delta}_1+s)\cdots(\bar{\delta}_k+s)} \\ \frac{-g(s)}{(\bar{\delta}_1+s)\cdots(\bar{\delta}_k+s)} & \frac{f(s)}{(\bar{\delta}_1+s)\cdots(\bar{\delta}_k+s)} \end{pmatrix} = \frac{1}{(\bar{\delta}_1 + s)\cdots(\bar{\delta}_k + s)} \begin{pmatrix} f^{*}(s) & g^{*}(s) \\ -g(s) & f(s) \end{pmatrix}.$$

Since $(A - DZ)^{(*)} + C^{(*)} = (U^{*}A - VZ)^{(*)} + C^{(*)}$ and noting that V is invertible in $\mathbf{H}^{\infty}$, we have

$$\mu_o = \min_{Z \in \mathbf{H}^{\infty}_{2\times 1}} \left\| \begin{pmatrix} U^{*}A - Z \\ C \end{pmatrix} \right\|_{\infty},$$

where $C \in \mathbf{H}^{\infty}$.

5.3.3 Transformation to an Operator Norm

In the following, the above infimal norm problem is transformed into an operator norm problem in a standard way using a well-known result of [36].

Let $\mathcal{T} : \mathbf{H}^2 \to \begin{pmatrix} (\mathbf{H}^2_{2\times 1})_- \\ \mathbf{H}^2 \end{pmatrix} \oplus \begin{pmatrix} \mathbf{H}^2_{2\times 1} \\ 0 \end{pmatrix}$ be defined as

$$\mathcal{T}f = \begin{pmatrix} U^{*}Af \\ Cf \end{pmatrix} = \begin{pmatrix} \Pi_{(\mathbf{H}^2_{2\times 1})_-} U^{*}Af \\ Cf \end{pmatrix} + \begin{pmatrix} \Pi_{\mathbf{H}^2_{2\times 1}} U^{*}Af \\ 0 \end{pmatrix}, \text{ for any } f \in \mathbf{H}^2.$$

Here $(\mathbf{H}^2_{2\times1})_- = \mathbf{L}^2_{2\times1} \ominus \mathbf{H}^2_{2\times1}$. Then we have [36, Theorem 1]

$$\mu_o = \|\mathcal{T}^* \mid_{\mathbf{K}_0} \|,$$

where $\mathbf{K}_0 = \begin{pmatrix} (\mathbf{H}^2_{2\times1})_- \\ \mathbf{H}^2 \end{pmatrix}$.

Let $\mathcal{C} = \Pi_{\mathbf{K}_0}\mathcal{T} : \mathbf{H}^2 \to \mathbf{K}_0$, then $\mu^2 = \|\mathcal{T}^* \mid_{\mathbf{K}_0} \|^2 = \|\mathcal{C}\mathcal{C}^*\| = \|\mathcal{C}^*\|^* = \|\mathcal{C}\|^2 = \|\mathcal{C}^*\mathcal{C}\|$.

Since $\mathcal{C} = \begin{pmatrix} \Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A \\ \mathbf{C} \end{pmatrix} : \mathbf{H}^2 \to \mathbf{K}_0, \mathcal{C}^* = \left((\Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A)^*, \mathbf{C}^* \right) : \mathbf{K}_0 \to \mathbf{H}^2$.

For any $x \in (\mathbf{H}^2_{2\times1})_-, y \in \mathbf{H}^2$,

$$< (\Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A)^* x, y > = < x, \Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A y > =$$

$$= < x, U^* A y > = < A^* U x, y > = < \Pi_{\mathbf{H}^2} A^* U x, y >$$

and for any $y, z \in \mathbf{H}^2$

$$< \mathbf{C}^* y, z > = < \Pi_{\mathbf{H}^2} C^* y, z >$$

so $(\Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A)^* = \Pi_{\mathbf{H}^2} A^* U$ and $\mathbf{C}^* = \Pi_{\mathbf{H}^2} C^*$ and thus

$$\begin{aligned} \mathcal{C}^*\mathcal{C} &= (\Pi_{\mathbf{H}^2} A^* U, \Pi_{\mathbf{H}^2} C^*) \begin{pmatrix} \Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A \\ \mathbf{C} \end{pmatrix} \\ &= \Pi_{\mathbf{H}^2} A^* U \Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A + \Pi_{\mathbf{H}^2} C^* C \end{aligned} \tag{5.18}$$

We conclude that

$$\mu_o^2 = \sup spec(\mathcal{C}^*\mathcal{C}) \tag{5.19}$$

Remark: By [30, Theorem 2], we know if $\mu_o > \|C\|_\infty$ then

$$\mu_o^2 = \sup spec(\mathcal{H}^*_{U^*A}\mathcal{H}_{U^*A} + \mathcal{T}_{C^*C}).$$

Here $\mathcal{H}_{U^*A}$ and $\mathcal{T}_{C^*C}$ denote Hankel and Toeplitz operators. It is easy to see that $\mathcal{H}^*_{U^*A} = \Pi_{\mathbf{H}^2} A^* U$, so (5.19) follows.

5.4 Computation of the Optimal Performance μ_o

From the previous section we know that the optimal performance μ_o is equal to the square root of the norm of the operator $\mathcal{C}^*\mathcal{C}$. In this section we shall show how to compute the discrete and essential spectra of this operator. From (5.19), these two computations allow us to find μ_o (as in [51] and [50]). Note that all the operations in theorems 1 and 2 are explicitly computable using the expressions in section 4.

5.4.1 Discrete Spectrum of $C^{(*)}$

Let $E = \begin{pmatrix} e^{-s\tau_s} & 0 \\ 0 & e^{-s\tau_v} \end{pmatrix}$, then $A = E^*A_1$, where A_1 is rational. Since in general $\tau_s \neq \tau_v$, the multiplication of E^* and A_1 is not commutative, so the techniques used in scalar case [50] to cancel the infinite dimentional parts of the plant no longer work for us. [35] assumes the following commutation condition:

$$E^*A_1 = \Omega_b M_b^*,$$

where Ω_b is rational and M_b is inner (possibly infinite dimensional), so that the cancellation of the infinite dimensional part works as it does for the scalar case. It is easy to see that this commutation condition can not be satisfied in the present case. The difficulty of the present problem, of course, is mainly due to the noncommutativity of the matrix multiplication.

Since the multiplication of E^* and A_1 is not commutative, for an eigenvector x_μ of $C^{(*)}$ associated with the eigenvalue μ^2, our idea here is to characterize A_1x_μ instead of x_μ as [50] does. The proof of the following lemma is basically the same as Lemma 1 in [50]. The rationality of A_1 and U allows us just to introduce an extra finite Blaschke product to "absorb" the unstable poles of $U^*A_1A_1^*$.

Lemma 7 *If μ^2 is an eigenvalue of $C^{(*)}$ and x_μ is an associated eigenfunction, then*

$$A_1x_\mu \in b_\mu b_\alpha d_e(\mathbf{H}^2_{2\times1})_-,$$

where

b_μ is a finite Blaschke product such that $b_\mu[(\mu^2 - C^{(*)})^{-1}]^* \in \mathbf{H}^\infty$;
b_α is a finite Blaschke product such that $b_\alpha U^*A_1A_1^* \in \mathbf{H}^\infty_{2\times2}$;
d_e is an inner function such that $d_eE^* \in \mathbf{H}^\infty_{2\times2}$.

Proof: Since $\mu^2x_\mu = C^{(*)}x_\mu$, we have

$$(\mu^2 - C^{(*)})x_\mu = (\Pi_{\mathbf{H}^2}A^*U\Pi_{(\mathbf{H}^2_{2\times1})_-}U^*A - \Pi_{\mathbf{H}^2_-}C^{(*)})x_\mu.$$

Multiply the above equation by $(\mu^2 - C^{(*)})^{-1}b_\mu^*b_\alpha^*d_e^*A_1$, we get

$$b_\mu^*b_\alpha^*d_e^*A_1x_\mu = (\mu^2 - C^{(*)})^{-1}b_\mu^*b_\alpha^*d_e^*A_1(\Pi_{\mathbf{H}^2}A^*U\Pi_{(\mathbf{H}^2_{2\times1})_-}U^*A - \Pi_{\mathbf{H}^2_-}C^{(*)})x_\mu \tag{5.20}$$

For any $h \in \mathbf{H}^2_{2\times1}$,

$$< b_\alpha^*d_e^*A_1(\Pi_{\mathbf{H}^2}A^*U\Pi_{(\mathbf{H}^2_{2\times1})_-}U^*A - \Pi_{\mathbf{H}^2_-}C^{(*)})x_\mu, h >=$$

$$=< (\Pi_{\mathbf{H}^2}A^*U\Pi_{(\mathbf{H}^2_{2\times1})_-}U^*A - \Pi_{\mathbf{H}^2_-}C^{(*)})x_\mu, b_\alpha d_e A_1^*h >=$$

$$=< A^*U\Pi_{(\mathbf{H}^2_{2\times1})_-}U^*Ax_\mu, b_\alpha d_e A_1^*h >=< \Pi_{(\mathbf{H}^2_{2\times1})_-}U^*Ax_\mu, U^*Ab_\alpha d_e A_1^*h >=$$

$$=< \Pi_{(\mathbf{H}^2_{2\times1})_-}U^*Ax_\mu, b_\alpha d_e U^*E^*A_1A_1^*h >= 0.$$

This means $b_\alpha^* d_\epsilon^* A_1(\Pi_{\mathbf{H}^2} A^* U \Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A - \Pi_{\mathbf{H}^2_-} C^{(*)}) x_\mu \in (\mathbf{H}^2_{2\times1})_-$. By the definition of b_μ and (5.20),

$$b_\mu^* b_\alpha^* d_\epsilon^* A_1 x_\mu \in (\mathbf{H}^2_{2\times1})_-,$$

therefore the lemma. ∎

Remark. b_μ, b_α and d_ϵ can be chosen as follows:

(1) b_μ can be chosen as the Blaschke product whose zeros are those zeros of $\mu^2 - C^{(*)}$ lying in $Re(s) > 0$.

(2) b_α can be chosen as the Blaschke product whose zeros are those poles of $U^* A_1 A_1^*$ lying in $Re(s) > 0$.

(3) d_ϵ can be chosen as $detE$ or $e^{-s\max(\tau_s,\tau_v)} = e^{-s\tau_v}$.

Note that we can rewrite (5.18) as

$$\begin{aligned}
C^{(*)} &= \Pi_{\mathbf{H}^2} A^* U (I - \Pi_{\mathbf{H}^2_{2\times1}}) U^* A + \Pi_{\mathbf{H}^2} C^{(*)} \\
&= \Pi_{\mathbf{H}^2}(A^{(*)} + C^{(*)}) - \Pi_{\mathbf{H}^2} A^* U \Pi_{\mathbf{H}^2_{2\times1}} U^* A \\
&= \Pi_{\mathbf{H}^2}(A^{(*)} + C^{(*)}) - (A^* U - \Pi_{\mathbf{H}^2_-} A^* U) \Pi_{\mathbf{H}^2_{2\times1}} U^* A \qquad (5.21)
\end{aligned}$$

It is not difficult to see that Lemma 7 does not provide enough information about the eigenspace of $C^{(*)}$ for us to conclude that $C^{(*)} - (A^{(*)} + C^{(*)})$ is of finite rank on the eigenspace because $e^{-s\tau_v}\mathbf{H}_- \subset e^{-s\tau_s}\mathbf{H}_-$. The following lemma provides us further information on the eigenspace so that we can show that $C^{(*)} - (A^{(*)} + C^{(*)})$ is indeed of finite rank on the eigenspace.

Lemma 8 *If x_μ is an eigenfunction of $C^{(*)}$, then*

$$x_\mu \in F(\mathbf{H}^2) + R_1 e^{-s\tau_s} \mathbf{H}^2_-,$$

where F is of finite rank, R_1 is rational.

Proof: Since $C^{(*)} x_\mu = \mu^2 x_\mu$, from (5.21) we have

$$\begin{aligned}
0 &= C^{(*)} x_\mu - \mu^2 x_\mu \\
&= (A^{(*)} + C^{(*)} - \mu^2) x_\mu - \Pi_{\mathbf{H}^2_-}(A^{(*)} + C^{(*)}) x_\mu - (A^* U - \Pi_{\mathbf{H}^2_-} A^* U) \Pi_{\mathbf{H}^2_{2\times1}} U^* A x_\mu \\
&= (A^{(*)} + C^{(*)} - \mu^2) x_\mu - \Pi_{\mathbf{H}^2_-}(A^{(*)} + C^{(*)}) x_\mu - A^* U \Pi_{\mathbf{H}^2_{2\times1}} U^* A x_\mu + \Pi_{\mathbf{H}^2_-} A^* U \Pi_{\mathbf{H}^2_{2\times1}} U^* A x_\mu \\
&\qquad (5.22)
\end{aligned}$$

Noting that both $\Pi_{\mathbf{H}^2_-}(A^{(*)} + C^{(*)}) \mid_{\mathbf{H}^2}$ and $\Pi_{\mathbf{H}^2_-} A^* U \mid_{\mathbf{H}^2}$ are of finite rank, we rewrite (5.22) as

$$0 = (A^{(*)} + C^{(*)} - \mu^2) x_\mu - A^* U \Pi_{\mathbf{H}^2_{2\times1}} U^* A x_\mu + F_1 \qquad (5.23)$$

where F_1 is of finite rank.

In Lemma 7, if we choose $d_\epsilon = e^{-s\tau_v}$, we have

$$\Pi_{\mathbf{H}^2_{2\times1}} U^* A x_\mu = \Pi_{\mathbf{H}^2_{2\times1}} U^* E^* A_1 x_\mu = \Pi_{\mathbf{H}^2_{2\times1}} U^* E^* b_\mu b_\alpha d_\epsilon \begin{pmatrix} h_{1-} \\ h_{2-} \end{pmatrix} =$$

$$= \Pi_{\mathbf{H}^2_{2\times1}} b_\mu b_\alpha U^* \begin{pmatrix} e^{-s\tau} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_{1-} \\ h_{2-} \end{pmatrix} = \Pi_{\mathbf{H}^2_{2\times1}} b_\mu b_\alpha U^* \begin{pmatrix} e^{-s\tau} h_{1-} \\ h_{2-} \end{pmatrix} =$$

$$= \Pi_{\mathbf{H}^2_{2\times1}} b_\mu b_\alpha U^* (\Pi_{\mathbf{H}^2_{2\times1}} + \Pi_{(\mathbf{H}^2_{2\times1})_-}) \begin{pmatrix} e^{-s\tau} h_{1-} \\ h_{2-} \end{pmatrix} =$$

$$= \Pi_{\mathbf{H}^2_{2\times1}} b_\mu b_\alpha U^* \begin{pmatrix} \Pi_{\mathbf{H}^2} e^{-s\tau} h_{1-} \\ 0 \end{pmatrix} + \Pi_{\mathbf{H}^2_{2\times1}} b_\mu b_\alpha U^* \Pi_{(\mathbf{H}^2_{2\times1})_-} \begin{pmatrix} e^{-s\tau} h_{1-} \\ h_{2-} \end{pmatrix} =$$

$$= b_\mu b_\alpha U^* \begin{pmatrix} \Pi_{\mathbf{H}^2} e^{-s\tau} h_{1-} \\ 0 \end{pmatrix} - \Pi_{(\mathbf{H}^2_{2\times1})_-} b_\mu b_\alpha U^* \begin{pmatrix} \Pi_{\mathbf{H}^2} e^{-s\tau} h_{1-} \\ 0 \end{pmatrix} +$$

$$+ \Pi_{\mathbf{H}^2_{2\times1}} b_\mu b_\alpha U^* \Pi_{(\mathbf{H}^2_{2\times1})_-} \begin{pmatrix} e^{-s\tau} h_{1-} \\ h_{2-} \end{pmatrix}.$$

Here $\tau = \tau_v - \tau_s$, $\begin{pmatrix} h_{1-} \\ h_{2-} \end{pmatrix} \in (\mathbf{H}^2_{2\times1})_-$.

Since both $\Pi_{(\mathbf{H}^2_{2\times1})_-} b_\mu b_\alpha U^* \,|_{\mathbf{H}^2_{2\times1}}$ and $\Pi_{\mathbf{H}^2_{2\times1}} b_\mu b_\alpha U^* \,|_{(\mathbf{H}^2_{2\times1})_-}$ have finite rank, we have

$$\Pi_{\mathbf{H}^2_{2\times1}} U^* A x_\mu = b_\mu b_\alpha U^* \begin{pmatrix} \Pi_{\mathbf{H}^2} e^{-s\tau} h_{1-} \\ 0 \end{pmatrix} + F_2 \tag{5.24}$$

where F_2 is of finite rank.

Combining (5.23) and (5.24) we have

$$\begin{aligned} 0 &= (A^{(\bullet)} + C^{(\bullet)} - \mu^2) x_\mu - b_\mu b_\alpha A^* \begin{pmatrix} \Pi_{\mathbf{H}^2} e^{-s\tau} h_{1-} \\ 0 \end{pmatrix} + F_3 \\ &= (A^{(\bullet)} + C^{(\bullet)} - \mu^2) x_\mu - b_\mu b_\alpha A_1^* E \begin{pmatrix} \Pi_{\mathbf{H}^2} e^{-s\tau} h_{1-} \\ 0 \end{pmatrix} + F_3 \\ &= (A^{(\bullet)} + C^{(\bullet)} - \mu^2) x_\mu - b_\mu b_\alpha a_{11}^* e^{-s\tau_s} \Pi_{\mathbf{H}^2} e^{-s\tau} h_{1-} + F_3 \end{aligned} \tag{5.25}$$

where $A_1^* \triangleq (a_{11}^*, a_{12}^*)$, $F_3 = F_1 - A^* U F_2$ is of finite rank.

Now from Lemma 7 we have $h_{1-} = b_\mu^* b_\alpha^* d_e^* a_{11} x_\mu$, so (5.25) becomes

$$\begin{aligned} 0 &= (A^{(\bullet)} + C^{(\bullet)} - \mu^2) x_\mu - b_\mu b_\alpha a_{11}^* e^{-s\tau_s} \Pi_{\mathbf{H}^2} e^{-s\tau} b_\mu^* b_\alpha^* d_e^* a_{11} x_\mu + F_3 \\ &= (A^{(\bullet)} + C^{(\bullet)} - \mu^2) x_\mu - b_\mu b_\alpha a_{11}^* e^{-s\tau_s} \Pi_{\mathbf{H}^2} b_\mu^* b_\alpha^* a_{11} e^{s\tau_s} x_\mu + F_3 \\ &= (A^{(\bullet)} + C^{(\bullet)} - \mu^2) x_\mu - a_{11}^{(\bullet)} x_\mu + b_\mu b_\alpha a_{11}^* e^{-s\tau_s} \Pi_{\mathbf{H}^2_-} b_\mu^* b_\alpha^* a_{11} e^{s\tau_s} x_\mu + F_3 \end{aligned} \tag{5.26}$$

or

$$x_\mu = (\mu^2 + a_{11}^{(\bullet)} - A^{(\bullet)} - C^{(\bullet)})^{-1} F_3 +$$
$$+ (\mu^2 + a_{11}^{(\bullet)} - A^{(\bullet)} - C^{(\bullet)})^{-1} b_\mu b_\alpha a_{11}^* e^{-s\tau_s} \Pi_{\mathbf{H}^2_-} b_\mu^* b_\alpha^* a_{11} e^{s\tau_s} x_\mu.$$

This completes the proof. ∎

Now from the second equality of (5.26) and Lemma 8 we get

$$\begin{aligned} (A^{(\bullet)} + C^{(\bullet)} - \mu^2) x_\mu &= b_\mu b_\alpha a_{11}^* e^{-s\tau_s} \Pi_{\mathbf{H}^2} b_\mu^* b_\alpha^* a_{11} e^{s\tau_s} (\mu^2 + a_{11}^{(\bullet)} - A^{(\bullet)} - C^{(\bullet)})^{-1} F_3 + \\ &\quad + b_\mu b_\alpha a_{11}^* e^{-s\tau_s} \Pi_{\mathbf{H}^2} a_{11}^{(\bullet)} (\mu^2 + a_{11}^{(\bullet)} - A^{(\bullet)} - C^{(\bullet)})^{-1} \Pi_{\mathbf{H}^2_-} b_\mu^* b^* \alpha a_{11} e^{s\tau_s} x_\mu + F_3 \end{aligned} \tag{5.27}$$

Noting that $\Pi_{\mathbf{H}^2} a_{11}^{(*)}(\mu^2 + a_{11}^{(*)} - A^{(*)} - C^{(*)})^{-1} |_{\mathbf{H}^2_-}$ is of finite rank, we see that actually the right hand side of (5.27) is of finite rank. So we have

Theorem 4 *Let* $\mathbf{E}$ *be the eigenspace of an eigenvalue* μ^2 *of* $C^{(*)}$*, then*

$$\Delta \triangleq C^{(*)} - (A^{(*)} + C^{(*)})$$

is of finite rank on $\mathbf{E}$.

Proof: From the above derivation we have that $A^{(*)} + C^{(*)} - C^{(*)}$ equals the right hand side of (19), which is of finite rank on $\mathbf{E}$. ∎

Now if μ^2 is an eigenvalue of $C^{(*)}$ and x_μ a corresponding eigenfunction, then

$$C^{(*)} x_\mu = \mu^2 x_\mu.$$

From Theorem 1, we have

$$[\mu^2 - (A^{(*)} + C^{(*)})] x_\mu = \Delta x_\mu.$$

Since the operator Δ is of finite rank, we can calculate the eigenvalue μ^2 and the corresponding eigenfunctions as in [50] and [14].

5.4.2 Essential Spectrum of $C^{(*)}$

In this section, we shall show how to compute the essential spectrum of $C^{(*)}$. We shall consider the following more general form of A:

$$A = \begin{pmatrix} m_1^* & 0 \\ 0 & m_2^* \end{pmatrix} A_1$$

where m_1 and m_2 are arbitrary inner functions and m_1 divides m_2 in $\mathbf{H}^\infty$. In fact, we have

Theorem 5 *The essential spectrum of* $C^{(*)}$ *is*

$$\begin{aligned} \sigma_e(C^{(*)}) \;=\; & \{A_1^{(*)}(j\omega) + C^{(*)}(j\omega) : j\omega \in \sigma_e(m_1)\} \cup \{\inf_\omega [a_{12}^{(*)}(j\omega) + C^{(*)}(j\omega)], \\ & \sup_\omega [a_{12}^{(*)}(j,\omega) + C^{(*)}(j\omega)]\} \cup \{\inf_\omega C^{(*)}(j,\omega), \sup_\omega C^{(*)}(j\omega)\} . \end{aligned}$$

Prior to proving this theorem, we give some preliminary results. We make the following decompositions

$$\begin{aligned} \mathbf{H}^2 &= \mathbf{H}_1 \oplus \mathbf{H}_2, \\ \mathbf{H}_1 &= \mathbf{H}_{11} \oplus \mathbf{H}_{12}, \end{aligned}$$

where $\mathbf{H}_2 = m_2 \mathbf{H}^2$, $\mathbf{H}_{11} = (m_1 \mathbf{H}^2)^\perp = (m_1 \mathbf{H}^2_-) \cap \mathbf{H}^2$. For any operators X, Y in a Hilbert space, by $X \sim Y$ we mean that $X - Y$ is compact. First, we shall prove some projection properties.

Proposition 5 *For any rational $R \in \mathbf{L}^\infty$, any inner functions m, m_3, and m divides m_2 in $\mathbf{H}^\infty$, m_2 divides m_3 in $\mathbf{H}^\infty$*

(a) $\Pi_{\mathbf{H}_2} mR|_{\mathbf{H}^2_-}$

(b) $\Pi_{\mathbf{H}_1} m_3 R|_{\mathbf{H}^2}$

are both finite rank operators.

Proof: Let $m_2 = m\,m_0$, since $\Pi_{\mathbf{H}^2} R|_{\mathbf{H}^2_-}$ is a finite rank operator, we have

$$\Pi_{\mathbf{H}_2} mR|_{\mathbf{H}^2_-} = \Pi_{\mathbf{H}_2} m_2(\Pi_{\mathbf{H}^2} + \Pi_{\mathbf{H}^2_-})Rm_0^*|_{\mathbf{H}^2_-} \sim \Pi_{\mathbf{H}_2} m_2 \Pi_{\mathbf{H}^2_-} Rm_0^*|_{\mathbf{H}^2_-} = 0 \ .$$

This proves (a).

Similarly, let $m_3 = m_2 m_4$. Since $\Pi_{\mathbf{H}^2_-} R|_{\mathbf{H}^2}$ is of finite rank, we have

$$\Pi_{\mathbf{H}_1} m_3 R|_{\mathbf{H}^2} = \Pi_{\mathbf{H}_1} m_2(\Pi_{\mathbf{H}^2} + \Pi_{\mathbf{H}^2_-})Rm_4|_{\mathbf{H}^2} \sim \Pi_{\mathbf{H}_1} m_2 \Pi_{\mathbf{H}^2} Rm_4|_{\mathbf{H}^2} = 0 \ .$$

This proves (b). ∎

Proposition 6 *For any rational $R \in \mathbf{L}^\infty$, the following operators*

$$\Pi_{\mathbf{H}_1} R|_{\mathbf{H}_2}, \quad \Pi_{\mathbf{H}_2} R|_{\mathbf{H}_1}, \quad \Pi_{\mathbf{H}_{11}} R|_{\mathbf{H}_{12}}, \quad \Pi_{\mathbf{H}_{12}} R|_{\mathbf{H}_{11}}$$

are of finite rank.

Proof: Since $\mathbf{H}_2 = m_2\mathbf{H}^2$ by Proposition 5(b)

$$\Pi_{\mathbf{H}_1} R|_{\mathbf{H}_2} = \Pi_{\mathbf{H}_1} Rm_2|_{\mathbf{H}_2}$$

is of finite rank.

Let $H = m_1\mathbf{H}^2$. Similarly we know that

$$\Pi_{\mathbf{H}_{11}} R|_H = \Pi_{\mathbf{H}_{11}} Rm_1|_{\mathbf{H}_2}$$

is of finite rank. Since $\mathbf{H}_{12} = H \cap \mathbf{H}_1 \subset H$, so $\Pi_{\mathbf{H}_{11}} R|_{\mathbf{H}_{12}}$ is of finite rank. By observing that

$$\Pi_{\mathbf{H}_2} R|_{\mathbf{H}_1} = (\Pi_{\mathbf{H}_1} R^*|_{\mathbf{H}_2})^* \ , \qquad \Pi_{\mathbf{H}_{12}} R|_{\mathbf{H}_{11}} = (\Pi_{\mathbf{H}_{11}} R^*|_{\mathbf{H}_{12}})^*$$

we also see that $\Pi_{\mathbf{H}_2} R|_{\mathbf{H}_1}$ and $\Pi_{\mathbf{H}_{12}} R|\mathbf{H}_{11}$ are of finite rank. ∎

The following lemma is a result in the proof of Theorem 1 of [50].

Lemma 9 *([50]) For a self-adjoint operator X on a Hilbert space $\mathbf{H}$, let $\mathbf{S}$ be a subspace of $\mathbf{H}$. If*

$$X\Pi_{\mathbf{S}} - \Pi_{\mathbf{S}} X \Pi_{\mathbf{S}}$$

is compact, then

$$\sigma_e(\Pi_{\mathbf{S}} X \Pi_{\mathbf{S}}) = \sigma_e(X) \ .$$

Proof of Theorem 2:

Let

$$\begin{aligned}\Delta_1 &\triangleq \Pi_{\mathbf{H}^2} A^* U \Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A,\\ \Delta_2 &\triangleq \Pi_{\mathbf{H}_1} A^* U_{P\mathbf{H}^2_{2\times1}} U^* A \Pi_{\mathbf{H}_1},\\ \Delta_3 &\triangleq \Pi_{\mathbf{H}_{12}} (u_{11}, u_{21}) a_{11}^* m_1 \Pi_{\mathbf{H}^2_{2\times1}} \binom{u_{11}^*}{u_{21}^*} a_{11} m_1^* \Pi_{\mathbf{H}_{12}}.\end{aligned}$$

Then

$$\begin{aligned}\Delta_1 &= \Pi_{\mathbf{H}^2} A^* U \Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A (\Pi_{\mathbf{H}_1} + \Pi_{\mathbf{H}_2}) \sim \Pi_{\mathbf{H}^2} A^* U \Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A \Pi_{\mathbf{H}_1} =\\ &= (\Pi_{\mathbf{H}_1} + \Pi_{\mathbf{H}_2}) A^* U \Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A \Pi_{\mathbf{H}_1}.\end{aligned}$$

By Proposition 5(a),

$$\Pi_{\mathbf{H}_2} A^* U \Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A \sim 0$$

and therefore

$$\begin{aligned}\Delta_1 &\sim \Pi_{\mathbf{H}_1} A^* U \Pi_{(\mathbf{H}^2_{2\times1})_-} U^* A \Pi_{\mathbf{H}_1}\\ &= \Pi_{\mathbf{H}_1} A_1^* A_1 \Pi_{\mathbf{H}_1} - \Delta_2 \end{aligned}\tag{5.28}$$

We now compute Δ_2. First we have

$$U^* A = \begin{pmatrix} u_{11}^* & u_{12}^* \\ u_{21}^* & u_{22}^* \end{pmatrix} \begin{pmatrix} m_1^* & 0 \\ 0 & m_2^* \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \begin{pmatrix} u_{11}^* \\ u_{21}^* \end{pmatrix} a_{11} m_1^* + \begin{pmatrix} u_{12}^* \\ u_{22}^* \end{pmatrix} a_{12} m_2^*$$

So

$$A^* U = (u_{11}, u_{21}) a_{11}^* m_1 + (u_{12}, u_{22}) a_{12}^* m_2 \, .$$

It is easy to see that $\Pi_{\mathbf{H}^2_{2\times1}} \binom{u_{12}^*}{u_{22}^*} a_{12} m_2^* \Pi_{\mathbf{H}_1}$ is of finite rank since $\binom{u_{12}^*}{u_{22}^*} a_{12}$ is rational. Also by Proposition 5(b), $\Pi_{\mathbf{H}_1} (u_{12}, u_{22}) a_{12}^* m_2 |_{\mathbf{H}^2_{2\times1}} \sim 0$. So we have

$$\begin{aligned}\Delta_2 &\sim \Pi_{\mathbf{H}_1} (u_{11}, u_{21}) a_{11}^* m_1 \Pi_{\mathbf{H}^2_{2\times1}} \begin{pmatrix} u_{11}^* \\ u_{21}^* \end{pmatrix} a_{11} m_1^* \Pi_{\mathbf{H}_1}\\ &= (\Pi_{\mathbf{H}_{11}} + \Pi_{\mathbf{H}_{12}}) (u_{11}, u_{21}) a_{11}^* m_1 \Pi_{\mathbf{H}^2_{2\times1}} \begin{pmatrix} u_{11}^* \\ u_{21}^* \end{pmatrix} a_{11} m_1^* (\Pi_{\mathbf{H}_{11}} + \Pi_{\mathbf{H}_{12}})\\ &\sim \Delta_3 \end{aligned}\tag{5.29}$$

The last equivalence $\sim$ is because that $\Pi_{\mathbf{H}^2_{2\times1}} \binom{u_{11}^*}{u_{21}^*} a_{11} m_1^* \Pi_{\mathbf{H}_{11}}$ is of finite rank and also by Proposition 5(b) $\Pi_{\mathbf{H}_{11}} (u_{11}, u_{21}) a_{11}^* m_1 |_{\mathbf{H}^2_{2\times1}} \sim 0$.

Further, since $\mathbf{H}_{12} \subset m_1 \mathbf{H}^2$, we see that $\Pi_{(\mathbf{H}^2_{2\times1})_-} \binom{u_{11}^*}{u_{21}^*} a_{11} m_1^* \Pi_{\mathbf{H}_{12}}$ is of finite rank, so

$$\begin{aligned}\Delta_3 &\sim \Pi_{\mathbf{H}_{12}} (u_{11}, u_{21}) a_{11}^* \begin{pmatrix} u_{11}^* \\ u_{21} \end{pmatrix} a_{11} \Pi_{\mathbf{H}_{12}} - \Pi_{\mathbf{H}_{12}} (u_{11}, u_{21}) a_{11}^* m_1 \Pi_{(\mathbf{H}^2_{2\times1})_-} \begin{pmatrix} u_{11}^* \\ u_{21}^* \end{pmatrix} a_{11} m_1^* \Pi_{\mathbf{H}_{12}}\\ &\sim \Pi_{\mathbf{H}_{12}} a_{11}^* a_{11} \Pi_{\mathbf{H}_{12}} \end{aligned}\tag{5.30}$$

here $u_{11}^{*}u_{11}+u_{21}^{*}u_{21}=1$ since $U^{*}U=I$.

Combining (5.18), (5.28), (5.29) and (5.30) we have

$$C^{(*)}=\Pi_{\mathbf{H}_1}A_1^{(*)}\Pi_{\mathbf{H}_1}-\Pi_{\mathbf{H}_{12}}a_{11}^{*}\Pi_{\mathbf{H}_{12}}+\Pi_{\mathbf{H}^2}C^{(*)}\ .$$

By Proposition 6, we further have

$$C^{(*)}=\Pi_{\mathbf{H}_{11}}(A_1^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{11}}+\Pi_{\mathbf{H}_{12}}(A_1^{(*)}+C^{(*)}-a_{11}^{(*)})\Pi_{\mathbf{H}_{12}}+\Pi_{\mathbf{H}_2}C^{(*)}\Pi_{\mathbf{H}_2}$$

and therefore

$$\sigma_e(C^{(*)})=\sigma_e(\Pi_{\mathbf{H}_{11}}(A_1^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{11}})\cup\sigma_e(\Pi_{\mathbf{H}_{12}}(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}})\cup\sigma_e(\Pi_{\mathbf{H}_2}C^{(*)}\Pi_{\mathbf{H}_2})\ .$$

Noting that

$$\begin{aligned}(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}}-\Pi_{\mathbf{H}_{12}}(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}}&=\Pi_{\mathbf{H}^2_-}(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}}+\\ &+(\Pi_{\mathbf{H}^2}-\Pi_{\mathbf{H}_{12}})(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}}\sim(\Pi_{\mathbf{H}^2}-\Pi_{\mathbf{H}_{12}})(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}}=\\ &=\Pi_{\mathbf{H}_{11}}(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}}+\Pi_{\mathbf{H}_2}(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}}\sim\Pi_{\mathbf{H}_2}(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}}\ .\end{aligned}$$

By Proposition 6, $\Pi_{\mathbf{H}_2}(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_1}\sim 0$ and since $\mathbf{H}_{12}\subset\mathbf{H}_1$, so

$$\Pi_{\mathbf{H}_2}(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}}\sim 0$$

and therefore

$$(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}}-\Pi_{\mathbf{H}_{12}}(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}}\sim 0\ .$$

So by Lemma 9, $\sigma_e(\Pi_{\mathbf{H}_{12}}(a_{12}^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{12}})=\sigma_e(a_{12}^{(*)}+C^{(*)})$.

Similarly, by Proposition 6,

$$\begin{aligned}&C^{(*)}\Pi_{\mathbf{H}_2}-\Pi_{\mathbf{H}_2}C^{(*)}\Pi_{\mathbf{H}_2}=\\ &\quad\Pi_{\mathbf{H}^2_-}C^{(*)}\Pi_{\mathbf{H}_2}+(\Pi_{\mathbf{H}^2}-\Pi_{\mathbf{H}_2})C^{(*)}\Pi_{\mathbf{H}_2}\sim\Pi_{\mathbf{H}_1}C^{(*)}\Pi_{\mathbf{H}_2}\sim 0\ .\end{aligned}$$

So by Lemma 9, $\sigma_e(\Pi_{\mathbf{H}_2}C^{(*)}\Pi_{\mathbf{H}_2})=\sigma_e(C^{(*)})$.

Note for any rational $R\in\mathbf{L}^\infty$, $R\Pi_{\mathbf{H}^2}=\Pi_{\mathbf{H}^2}R\Pi_{\mathbf{H}^2}+\Pi_{\mathbf{H}^2_-}R\Pi_{\mathbf{H}^2}\sim\Pi_{\mathbf{H}^2}R\Pi_{\mathbf{H}^2}$. So $\sigma_e(R)=\sigma_e(\Pi_{\mathbf{H}^2}R)$. By [38, Theorem 6.2 (i), p. 55] we have

$$\begin{aligned}\sigma_e(a_{12}^{(*)}+C^{(*)})&=\{\inf_\omega[a_{12}^{(*)}(j\omega)+C^{(*)}(j\omega)],\ \sup_\omega[a_{12}^{(*)}(j\omega)+C^{(*)}(j\omega]\}\ ,\\ \sigma_e(C^{(*)})&=\{\inf_\omega C^{(*)}(j\omega),\ \sup_\omega C^{(*)}(j\omega)\}\ .\end{aligned}$$

By [33, Corollary 1, p.125] we have

$$\sigma_e(\Pi_{\mathbf{H}_{11}}(A_1^{(*)}+C^{(*)})\Pi_{\mathbf{H}_{11}})=\{A_1^{(*)}(j\omega)+C^{(*)}(j\omega):j\omega\in\sigma_e(m_1)\}\ .$$

■

5.5 Optimal Compensators

After obtaining the optimal performance μ_o, one can compute the suboptimal compensators using the rational approximation techniques. In the multivariate case, even for rational plants, there is more than one optimal compensator in general [22]. However, in the present case, we observe that our problem is a "vector" problem rather than a general "matrix" problem, and the operator we get in the previous section is a scalar operator. This makes it possible for us to find an eigenfunction corresponding to the largest eigenvalue μ_o of the operator, and then we can find the optimal compensator as in the scalar case.

Assume that $\mu_o > \|C\|_\infty$. Using the technique described in [30], we can transfer the original mixed sensitivity problem into the following pure sensitivity problem (with the optimal performance $\mu_o' = 1$):

$$\min_{\tilde{Z}\in \mathbf{H}^\infty} \|H - \tilde{Z}\|_\infty = 1,$$

where $M^{(\bullet)} \triangleq \mu_o^2 - C^{(\bullet)}, M, M^{-1} \in \mathbf{H}^\infty, H \triangleq U^\bullet A M^{-1} \in \mathbf{L}^\infty$.

This is a infinite dimensional pure sensitivity problem with two different time delays. The method developed in the previous section can be used to find the maximal eigenfunctions (corresponding to the largest eigenvalue $\mu_o' = 1$) of the corresponding Hankel operator and the minimal sensitivity can be found as described in [51].

5.6 Conclusion

In this chapter we have shown how to compute the infimal mixed sensitivity for the full infinite dimensional model of multivariable plant considered in [46], without resorting to finite dimensional approximation. We are presently preparing to compute numerical values for comparison with the results in [46]. The procedure presented here is readily generalizable to a class of MIMO plants [47].

Chapter 6

Numerical Inner/Outer Factorization

6.1 Introduction

The motivation for the developments in this chapter is interest in explicitly computing solutions to H^∞-optimal control problems for distributed parameter systems, such as the damped flexible beam in Section 2.1. In this case one has an irrational transfer function

$$P(s) = N(s)/D(s) \tag{6.1}$$

where N and D are both functions in H^∞. The solution to the mixed sensitivity problem [50],[15] involves finding the largest solution μ_0 on an interval $[a, b] \subset \mathbf{R}$ to an equation of the form

$$N_i(j\zeta(\mu))R_1(\mu) = N_i(-j\zeta(\mu))R_2(\mu) \tag{6.2}$$

where N_i is the inner factor of N (see Example 1 below for further details), and ζ, R_1, and R_2 are certain functions of μ. (μ_0 is the largest eigenvalue of an operator which is related to the mixed sensitivity operator. See [50].) If we search for μ_0 by numerically computing solutions to this equation, computation of the infimal values for H^∞-control criteria will involve the numerical evaluation of the term $N_i(j\omega)$ for different values of ω.

For general transfer functions, one will not be able to explicitly write down an inner-outer factorization, and so the evaluation of the term $N_i(j\omega)$ is not simply a matter of substituting a number into a formula. In this chapter we present a numerical technique for this evaluation without explicitly finding the inner factor.

We present three examples of irrational inner factors on which we will test our numerical technique. In Example 1 below, the inner factor of the transfer function is an infinite Blaschke product. In this case we can explicitly find the right half plane zeros, but there are infinitely many. For Example 2, the transfer function has a singular inner factor, yet an explicit factorization and formula is available. In Example 3, the inner factor appears to be an infinite Blaschke product, but the alternative of taking a continued product of separately computed zeros is not attractive since the implicit nature of the expression for the transfer function and lack of further information on the location of the right half plane zeros make their computation difficult.

6.2 Mathematical Framework

The outer factor of an H^∞ function is given in the open right half plane by [26, p. 133]

$$f_o(s) = \exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty} \log|f(it)| \cdot \frac{ts+i}{t+is}\frac{dt}{1+t^2}\right].$$

On the imaginary axis the boundary value function of f_o exists almost everywhere and is equal to

$$\lim_{Re(s)\to 0} \exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty} \log|f(it)| \cdot \frac{ts+i}{t+is}\frac{dt}{1+t^2}\right].$$

Taking $s = \sigma + i\omega$, with σ, $\omega \in \mathbf{R}$, we have

$$f_o(i\omega) = \lim_{\sigma\to 0}\exp\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\left(\frac{\sigma(1+t^2)}{(\omega-t)^2+\sigma^2}\right) - i\left(t + \frac{(\omega-t)(1+t^2)}{(\omega-t)^2+\sigma^2}\right)\right]\log|f(it)|\frac{dt}{1+t^2}\right\}.$$

We know the limit of the real part of the integral, since

$$|f_o(i\omega)| = |f(i\omega)| \quad \text{a.e.}$$

The limit of the imaginary part of the integral in the last expression is defined as an integral on the imaginary axis in the sense of a Cauchy principal value, which we can simplify as follows:

$$\begin{aligned}
\arg(f_o(i\omega)) &= \lim_{\sigma\to 0}\left\{-\frac{1}{\pi}\int_{-\infty}^{\infty}\left(t+\frac{(\omega-t)(1+t^2)}{(\omega-t)^2+\sigma^2}\right)\log|f(it)|\frac{dt}{1+t^2}\right\} \\
&= -\frac{1}{\pi}⨍_{-\infty}^{\infty}\frac{\omega t+1}{(1+t^2)(\omega-t)}\log|f(it)|dt \\
&= -\frac{1}{\pi}\left\{⨍_{-\infty}^{\infty}\frac{\omega t+1}{(1+t^2)(\omega-t)}\log\left|\frac{f(it)}{f(i\omega)}\right|dt + \log|f(i\omega)|⨍_{-\infty}^{\infty}\frac{\omega t+1}{(1+t^2)(\omega-t)}dt\right\}
\end{aligned}$$

Considering the last term, we have:

$$\begin{aligned}
⨍_{-\infty}^{\infty}\frac{\omega t+1}{(1+t^2)(\omega-t)}dt &= ⨍_{-\infty}^{\infty}\left(\frac{t}{1+t^2}+\frac{1}{\omega-t}\right)dt \\
&= ⨍_{-\infty}^{\infty}\frac{t}{1+t^2}dt + \lim_{R\to\infty}\lim_{\epsilon\to 0}\left(\int_{\omega+\epsilon}^{R}\frac{dt}{\omega-t}+\int_{-R}^{\omega-\epsilon}\frac{dt}{\omega-t}+⨍_{\omega-\epsilon}^{\omega+\epsilon}\frac{dt}{\omega-t}\right) \\
&= ⨍_{-\infty}^{\infty}\frac{d(t^2)}{1+t^2} + \lim_{R\to\infty}\lim_{\epsilon\to 0}\left\{-\int_{\omega+\epsilon}^{R}\frac{dt}{t-\omega} - \int_{R}^{\epsilon-\omega}\frac{du}{\omega+u} + ⨍_{-\epsilon}^{\epsilon}\frac{dv}{v}\right\} \\
&= 0 + \lim_{R\to\infty}\lim_{\epsilon\to 0}\left\{-\log(t-\omega)\Big|_{\omega+\epsilon}^{R} - \log(\omega+u)\Big|_{R}^{\epsilon-\omega} + ⨍_{-\epsilon}^{\epsilon}\frac{dt}{t}\right)\} \\
&= \lim_{R\to\infty}\lim_{\epsilon\to 0}\left\{\log\left(\frac{R+\omega}{R-\omega}\right) + ⨍_{-\epsilon}^{\epsilon}\frac{dt}{t}\right\} \\
&= 0.
\end{aligned}$$

Now we note that

$$\frac{\omega t+1}{\omega-t}\cdot\log\left|\frac{f(it)}{f(i\omega)}\right|$$

has a removable singularity at $t=\omega$, and that since

$$\frac{\log|f(it)|}{1+t^2}\in L^1$$

(because $f(s)\in H^\infty$ [26, p. 51]),

$$\frac{\omega t+1}{(1+t^2)(\omega-t)}\log\left|\frac{f(it)}{f(i\omega)}\right|\in L^1.$$

Thus

$$\int_{-\infty}^{\infty}\frac{\omega t+1}{(1+t^2)(\omega-t)}\log\left|\frac{f(it)}{f(i\omega)}\right|dt$$

is finite as an ordinary (Lebesgue) integral. Therefore, we can compute

$$\arg(f_o(i\omega))=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\omega t+1}{(1+t^2)(\omega-t)}\log\left|\frac{f(it)}{f(i\omega)}\right|dt$$

by evaluating the integral numerically as an ordinary integral.

6.3 Discussion of Numerical Techniques

Four problems arise in the numerical computation:

1. the integration has infinite range,
2. the integrand has a removable singularities at $t=\omega$, and possibly other points which may be due to the formula for the transfer function,
3. the integrand will have integrable singularities at the (possibly non-rational) zeros of $f(i\omega)$, and
4. the presence of the logarithm in the kernel may lead to intrinsic computer word-length problems.

The first three are standard problems in numerical quadrature, and our approach is to assemble standard solutions, as we discuss below. The fourth problem is more serious, as we discuss in some detail below.

To complete the numerical computation of the inner factor of f, After finding the numerical value of $\arg(f_o(i\omega))$, we can simply compute

$$\begin{aligned}f_i(i\omega) &= \frac{f(i\omega)}{|f(i\omega)|\exp(i\cdot\arg(f_o(i\omega))}\\ &= \frac{f(i\omega)}{|f(i\omega)|}\cdot\exp[-i\cdot\arg(f_o(i\omega))]\end{aligned}$$

We now discuss the numerical problems mentioned above, and describe our approach in their solution. The basic reference we have used for a discussion of the issues and techniques in quadrature is [37]. (For the actual computer programs we obtained updated versions of the programs in [37] via electronic mail ("netlib") from Oak Ridge National Laboratories.)

1. Infinite range of integration. There are basically two standard approaches: either (a) transform the range to a finite interval, or else (*b*) approximate the infinite integral by integration over a finite range. Method (*b*) is recommended in [37, p. 80] for the case in which the integrand decays "rapidly" to zero at ∞, and this is the method we have adopted. We remark that we have also tried method (a), and this approach can generate an additional problem: if the integrand does not decay sufficiently quickly at ∞, then when transformed to the finite interval the integrand may be unbounded at the image of ∞, leading to inefficient stepsize selection for an adaptive quadrature program. We encountered this problem in Example 1 discussed below, and the difficulty is compounded by the wordlength problems in evaluating the integrand in the vicinity of ∞, which we discuss further below.

2. Removable singularities. For finite removable singularities this is easily handled by using a polynomial approximation in a neighborhood of the singularity. In our Example 1, we have a removable singularity at 0 due to the expression for the transfer function. Here we used a linear approximation to the integrand over a neighborhood which yields a variation in the function two orders of magnitude less than the estimated absolute accuracy requested of the adaptive quadrature program. For the singularity at ω one could use a similar approach, although in our tests we simply avoided values of ω which were likely abscissae in the quadrature (by taking logarithmically spaced points at which to evaluate the outer factor).

3. Integrable singularities. Since the integrand is in L^1, of course all singularities on the imaginary axis will be integrable. So far we have only considered an example with an irrational zero at ∞. We expect most finite zeros to be easier to treat, for the following reason: A finite zero of the transfer function will lead to an unbounded but integrable integrand at the zero, due to the *log* function. Such singularities are treated in [37]. The idea is to break up the interval of integration into subintervals, with the origin as an endpoint, and then use a specialized routine.

In the case of the zero at ∞, integrand must vanish at infinity, so this behaviour occurs by means of the quotient of two large numbers vanishing. This can, and does, in Example 1, lead to intrinsic numerical problems, as we discuss next.

4. For large t, the integrand essentially looks like $\frac{\log|f(it)|}{1+t^2}$. Assuming f has a zero at infinity, the numerator becomes unbounded (and negative) for large t, and the integrand is a quotient of two large numbers. Our ability to integrate this numerically depends upon a critical balance between two effects: on the one hand the quotient is getting small, while on the other hand the denominator is getting large. If the rate of increase of the numerator is sufficiently less than that of the denominator, then we can start neglecting the integrand for numerical purposes (setting it to be zero overall) before the size of the denominator exceeds the value allowed by machine wordlength. In Example 1, we see a case in which this balance is at a marginally acceptable level. In the other examples, there is no problem.

6.4 Examples

Example 1. Our first example is a transfer function model for the damped Euler-Bernoulli model presented in Section 2.1. We performed the computations for this model, and compared the accuracy against the phase of the inner factor computed by forming the Blaschke product from the first 100 right-half-plane zeros. (By adding zeros, it is easy to check the latter estimate of the phase has essentially no error.) In Figure 6.1 we plot the error in computed phase of the inner factor as a function of frequency.

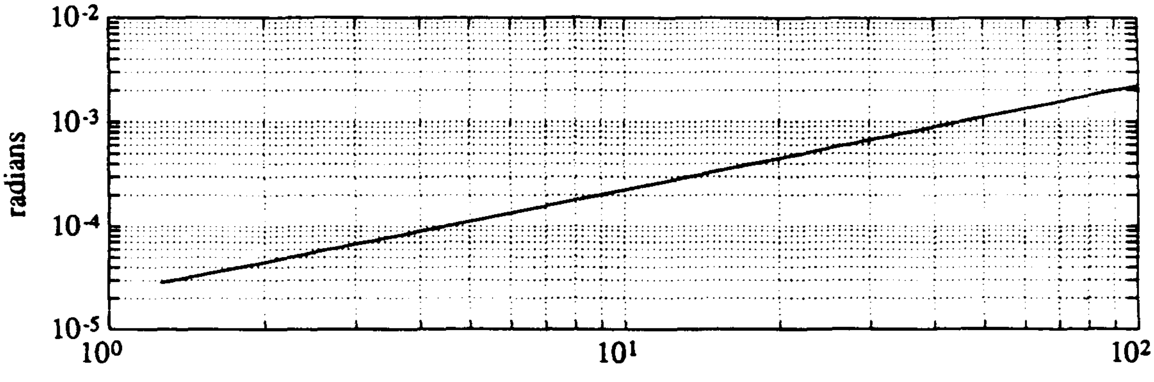

Figure 6.1: Error in inner factor phase for beam.

In order to attempt to reduce the error in the quadrature procedure it is necessary to examine the source of the error. The infinite integral is approximated using a finite interval, so there is an error contribution from the "tails" of the integral. Since the integrand is smooth, the adaptive quadrature routine easily computes the integral over the finite range, so long as the integrand can be evaluated. Using the asymptotic behavior (2.23) above, the tail looks like

$$\Delta \arg(f_o(i\omega)) \approx -\frac{1}{\pi}\left[\left(\int_R^{\infty} + \int_{-\infty}^{-R}\right)\frac{\omega}{t^2}\left(-\log\left|\frac{t}{\omega}\right| - 9.2\left(\sqrt[4]{|\,t\,|} - \sqrt[4]{|\,\omega\,|}\right)\right)dt\right]^{-1} \quad (6.3)$$

$$\approx -\frac{2}{\pi}\left[\int_R^{\infty}\frac{\omega}{t^2}\left(-9.2\sqrt[4]{|\,t\,|}\right)dt\right]^{-1} \quad (6.4)$$

which explains the linear error behavior seen in Figure 6.1. From the expression (6.4), it is clear (as it is intuitively) that we should lengthen the interval of integration in order to reduce the error.

Unfortunately, the interval $[-10^8, 10^8]$ used in this computation is roughly the largest interval we can use with the double precision computer program employed here, without replacing the integrand by an approximation. The reason is that the smallest non-zero number representable on the computer used is roughly 2×10^{-308}. Using the approximation (2.23), we see that at roughly $\omega = 3 \times 10^7$ the magnitude of the transfer function saturates machine precision. In practice the problem manifests itself not as an underflow but as an error in the compiler's logarithm computation, when the program is unable to normalize a floating point number due to a saturated exponent.

Actually, the naïve approach to computing the integrand has problems with an even smaller quadrature interval, since, in the formula (6.1) for the transfer function, the denominator is roughly the square of reciprocal of the transfer function at high frequency. It is only by carefull decomposition and normalization of the integrand that we are able to attain the full interval. We have not explored the use of the asymptotic behavior (2.23) as a direct approximation in order to extend the interval of integration.

Example 2. Consider the case of a pure time delay cascaded with a rational outer factor which is non-zero on the imaginary axis.

$$T(s) = \frac{1}{s+1}e^{-s}$$

In this case we can directly find the phase of the inner factor, and in fact the quadrature technique can also attain an essentially perfect result.

Example 3. We consider the transfer function of Section 2.2 of a flexible beam having a tip mass at one end and a motor at the other, taking into account both bending and torsion. Inspection of equations (2.45 - 2.47) quickly leads one to conclude that an explicit inner-outer factorization will not be possible. We have not yet performed computations for this model, but it seems clear that there is little hope checking the results by taking a finite Blacshke product.

6.5 Conclusions

We have presented a numerical technique for the computation of the inner factor of a stable transfer function, intended for use in solving $\mathbf{H}^\infty$ problems when an explicit factorization is not available. Preliminary computational results show that the technique works, although it is also clear that when the zeros of an infinite Blaschke product are readily computable, this is not the method of choice. Work on applying this method to the beam in Example 3, among other models, is in progress, along with the integration of this computation into a solution for the overall problem addressed by (6.2).

Chapter 7

Conclusions

We believe that our work to date has shown that there is a reasonable prospect that direct $\mathbf{H}^\infty$ design of feedback controls for distributed parameter systems will develop into a practical procedure. The work reported here shows some of the potential, both for computing actual designs and for finding the limits of attainable performance. We have also been able to obtain some insight into the nature of the optimality criteria considered by examining exact solutions. Obviously much remains to be done, in terms of increasing generality of the theory (to cover, for example, more complex SISO systems, MIMO systems, and more realistic and comprehensive design criteria) and in terms of developing numerical procedures for computing solutions.

Appendix A

Some Projection Formulas

Here we shall prove some projection formulas used in Chapter 4. These projection formulas show how to compute the projections of Π_+ and Π_- for some classes of $\mathbf{L}^2$ functions by means of residue theorem.

We shall use $\mathbf{C}$ to denote the complex plane, $\mathbf{C}^+$ and $\mathbf{C}^-$ to denote the open right and left half planes. $\mathbf{H}^2, \mathbf{H}^2_-, \mathbf{H}^\infty$ and $\mathbf{H}^\infty_-$ are Hardy spaces on the half plane.

Lemma 10 *For a rational function $Q(s)$ on $\mathbf{C}$ with $Q(j\omega) \in \mathbf{L}^\infty(j\mathbf{R})$,*

$$|Q(x+j\omega) - Q(j\omega)| \to 0 \text{ uniformly in } \omega$$

as $x \to 0$.

Proof: Since Q is proper, $\lim_{s\to\infty} Q(s) = q$ exists. So for any $\varepsilon > 0$, there exists an $L > 0$ such that for $|s| > L$,

$$|Q(s) - q| < \frac{\varepsilon}{2}.$$

So for $|\omega| > L$, we have

$$|Q(x+j\omega) - Q(j\omega)| \leq |Q(x+j\omega) - q| + |Q(j\omega) - q| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

Since $Q \in \mathbf{L}^\infty(j\mathbf{R}), Q$ has no poles on $j\mathbf{R}$, so it is continuous on $j\mathbf{R}$. For sufficiently small $\eta > 0$, $Q(s)$ is continuous on $[-\eta, \eta] \times [-jL, jL]$, so it is uniformly continuous on this closed rectangle. Therefore there exists a $\delta > 0$ such that for $|x| = |(x+j\omega) - j\omega| < \delta$ we have

$$|Q(x+j\omega) - Q(j\omega)| < \varepsilon.$$

∎

Lemma 11 *For Q as in Lemma 10 and $g \in \mathbf{H}^1_-$,*

$$\lim_{x\to 0^-} \int_{-\infty}^{\infty} |(Qg)(x+j\omega) - (Qg)(j\omega)| d\omega = 0.$$

Proof: Note

$$\int_{-\infty}^{\infty} |(Qg)(x+j\omega)-(Qg)(j\omega)|d\omega \le \int_{-\infty}^{\infty} |Q(x+j\omega)||g(x+j\omega)-g(j\omega)|d\omega +$$

$$+\int_{-\infty}^{\infty} |g(j\omega)||Q(x+j\omega)-Q(j\omega)|d\omega \le \|Q\|_{\mathbf{L}^\infty} \int_{-\infty}^{\infty} |g(x+j\omega)-g(j\omega)|d\omega +$$

$$+\sup_{\omega} |Q(x+j\omega)-Q(j\omega)| \int_{-\infty}^{\infty} |g(j\omega)|d\omega = \{1\}+\{2\}.$$

The first term tends to 0 by a result for $\mathbf{H}^p$ spaces [26, p. 128]. The second term tends to 0 by Lemma 10. ∎

Lemma 12 *For* $f(j\omega) \in \mathbf{L}^2$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(j\omega)}{j\omega - s} d\omega = \begin{cases} (\Pi_+ f)(s), s \in \mathbf{C}^+; \\ (\Pi_- f)(s), s \in \mathbf{C}^-. \end{cases}$$

Proof: See [7, p.195]. ∎

Formula A1: For a function $f = Qh_-$ with $Q \in R\mathbf{H}^\infty$ and $h_- \in \mathbf{H}^2_-$,

$$(\Pi_+ f)(s) = \frac{1}{2\pi j} \oint_\gamma \frac{f(z)}{s-z} dz \quad \text{for} \quad s \in \mathbf{C}^+,$$

where γ is a contour in $\mathbf{C}^-$ and encircles the poles of Q.
Proof: From Lemma 12 we only need to show $\oint_\gamma \frac{f(z)}{s-z}dz = j\int_{-\infty}^{\infty} \frac{f(j\omega)}{s-j\omega}d\omega$.

Let A be the region encircled by γ. Since $\frac{f(z)}{s-z}$ is analytic on $\mathbf{C}^- \setminus A$ for $s \in \mathbf{C}^+$, we see for any $R > 0$ sufficiently large and $\delta > 0$ sufficiently small,

$$\oint_\gamma \frac{f(z)}{s-z}dz = \oint_{\Gamma_{R,\delta}} \frac{f(z)}{s-z}dz = j\int_{-R}^{R} \frac{f(-\delta+j\omega)}{s-(-\delta+j\omega)}d\omega + j\int_\alpha^\beta \frac{f(Re^{j\theta})}{s-Re^{j\theta}} Re^{j\theta} d\theta,$$

where $\frac{\pi}{2} < \alpha < \beta < \frac{3\pi}{2}$.

For R large enough we have $|\frac{s}{Re^{j\theta}}| < \frac{1}{2}$. We know $h_-(Re^{j\theta}) \to 0$ uniformly as $R \to \infty$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ [26, p.125]. So

$$|j\int_\alpha^\beta \frac{f(Re^{j\theta})}{s-Re^{j\theta}} Re^{j\theta} d\theta| \le \|Q\|_{\mathbf{L}^\infty} \int_\alpha^\beta |\frac{h_-(Re^{j\theta})}{\frac{s}{Re^{j\theta}}-1}| d\theta \le 2\|Q\|_{\mathbf{L}^\infty} \int_\alpha^\beta |h_-(Re^{j\theta})| d\theta \to 0$$

as $R \to \infty$. So we have

$$\oint_\gamma \frac{f(z)}{s-z}dz = j\int_{-\infty}^{\infty} \frac{f(-\delta+j\omega)}{s-(-\delta+j\omega)}d\omega.$$

Also notice that for $h_- \in \mathbf{H}^2$, $h_-(j\omega) \in \mathbf{L}^2$, so $\frac{h_-(j\omega)}{s-j\omega} \in \mathbf{L}^1$. By Lemma 11,

$$\lim_{\delta \to 0} \int_{-\infty}^{\infty} \frac{f(-\delta+j\omega)}{s-(-\delta+j\omega)}d\omega = \int_{-\infty}^{\infty} \frac{f(j\omega)}{s-j\omega}d\omega.$$

Therefore

$$\oint_\gamma \frac{f(z)}{s-z}dz = j\int_{-\infty}^{\infty} \frac{f(j\omega)}{s-j\omega}d\omega.$$

This proves Formula (A1). ∎

By Formula A1 and the residue theorem, we have

Formula A1′: If Q has poles $p_1, p_2, \ldots, p_n$ in $\mathbf{C}^-$, then

$$(\Pi_+ f)(s) = \frac{1}{2\pi j}\oint_\gamma \frac{f(z)}{s-z}dz = \sum_{i=1}^{n} \frac{R^Q(p_i)h_-(p_i)}{s-p_i},$$

where $R^Q(p_i)$ denotes the residue of Q at p_i.

Similarly, we have corresponding formulas for a function $f = Qh_+$ with $Q \in R\mathbf{H}^\infty_-$ and $h_+ \in \mathbf{H}^2$. We state them explicitly as follows:

Formula A2: For a function $f = Qh_+$ with $Q \in R\mathbf{H}^\infty_-$ and $h_+ \in \mathbf{H}^2$,

$$(\Pi_- f)(s) = \frac{1}{2\pi j}\oint_\gamma \frac{f(z)}{s-z}dz \quad \text{for} \quad s \in \mathbf{C}^-,$$

where γ is a contour in $\mathbf{C}^+$ and encircles the poles of Q.

Formula A2′: If Q has poles $p_1, p_2, \ldots, p_n$ in $\mathbf{C}^+$, then

$$(\Pi_- f)(s) = \frac{1}{2\pi j}\oint_\gamma \frac{f(z)}{s-z}dz = \sum_{i=1}^{n} \frac{R^Q(p_i)h_-(p_i)}{s-p_i},$$

where $R^Q(p_i)$ denotes the residue of Q at p_i.

Appendix B

Computation of the Eigenvalues and Eigenfunctions of $\mathcal{T}$

In this appendix, we shall use the projection formulas developed in Appendix A to provide explicit formulas for the computation of the eigenvalues and eigenfunctions of $\mathcal{T}$.

Recall that for an eigenvalue μ^2 and an associated eigenfunction x_μ we have

$$(\mu^2 - W_1^* W_1)x_\mu = \Delta_\mu x_\mu \tag{B.1}$$

and

$$\Delta_\mu = \Pi_- A^*(\Pi_+ A - \Pi_- B) - A^* \Pi_+ A - \Pi_- W_1^* W_1 + A^* \Pi_- B + \Pi_+ B^* [\Pi_-(A + B)] \tag{B.2}$$

where

$$A = \frac{W_1^* W_1}{W^*} N_i^* D_i^*,$$
$$B = -W N_o X D_i^*,$$
$$W^* W = W_1^* W_1 + W_2^* W_2.$$

Further we explicitly write out W_1 and D_i as follows

$$W_1 = \frac{F(s)}{(s + w_1)\cdots(s + w_k)} \in \mathbf{H}^\infty,$$

where $F(s)$ is a polynomial in s, $w_j \in \mathbf{C}^+, j = 1, ..., k$, and

$$D_i = \frac{(\bar{d}_1 - s)\cdots(\bar{d}_n - s)}{(d_1 + s)\cdots(d_n + s)},$$

where $d_j \in \mathbf{C}^+, j = 1, ..., n$.

Remark 27 *1. By the definition of A, it is easy to see that the left half plane poles of A are $\{-w_1, ..., -w_k\} = \mathcal{P}(W_1)$.*

2. Since B_W is a Blaschke product such that

$$B_W \frac{W_1^*}{W^*} \in \mathbf{H}^\infty$$

and B_μ is a Blaschke product such that

$$B_\mu(\mu^2 - F^* F)^{-1} \in \mathbf{H}^\infty,$$

order(B_W) = *number of poles of* $\frac{W_1^*}{W^*}$ *on the right half plane* = order(B_μ). *We see that*

$$\text{order}(B_W) = \text{order}(B_\mu).$$

3. By the definition of B_W, we know that the right half plane poles of $\frac{W_1^*}{W^*}$ are $\{-\bar{p}_j^{B_W} : j = 1, ..., m\}$, or equivalently, the left half plane poles of $\frac{W_1}{W}$ are $\{p_j^{B_W} : j = 1, ..., m\}$.

There are five terms in (B.2). By Proposition 2, $R^{x_\mu}(p_j^{B_W}) = 0, j = 1, ..., m$. In the generic case, we write down the basis functions for the range of each term as an operator on $\mathbf{K}_\mu$ as follows:

1st term:

$$\frac{1}{s - w_1}, ..., \frac{1}{s - w_k}.$$

2nd term: (i)

$$A^* \frac{1}{s + w_1}, ..., A^* \frac{1}{s + w_k}.$$

(ii)

$$A^* \frac{1}{s - p_j^{B_\mu}}, j = 1, 2, ..., m.$$

3rd term:

$$\frac{1}{s - w_1}, ..., \frac{1}{s - w_k}.$$

4th term:

$$A^* \frac{1}{s - \bar{d}_1}, ..., A^* \frac{1}{s - \bar{d}_n}.$$

5th term:

$$\frac{1}{s + d_1}, ..., \frac{1}{s + d_n}.$$

We shall use $\phi_j, j = 1, ..., 2k + m + 2n$ to denote the above $2k + m + 2n$ basis functions of Δ_μ.

So we see that the rank of Δ_μ is

$$rank(\Delta_\mu) = 2k + m + 2n = 2[\text{order}(W_1) + \text{order}(D_i)] + \text{order}(B_\mu).$$

Remark 28 *In [50, Lemma 3], the authors consider the stable plant case and state that* $rank(\Delta_\mu) = 2[\text{order}(G) + \text{order}(F)]$ *in the generic case. In the stable plant case* G *reduces to* $\frac{W_1^* W_1}{W^*}$. *In fact in [50]* G *is also assumed to be in* $\mathbf{H}^\infty$. *Therefore, in that context* $B_W G$ *is substituted for* G, *and* $B_W M$ *for* M. *Of course,* $B_W G$ *has the same order as* G, *which is* $\text{order}(W^{-1}) = \text{order}(F)$. *By the definition of* B_μ, $B_\mu(\mu - F^* F)^{-1} \in \mathbf{H}^\infty$, *so* $\text{order}(F) = \text{order}(B_\mu)$. *But we show that* $rank(\Delta_\mu) = 2\text{order}(W_1) + \text{order}(B_\mu)$. *(In the stable plant case* $D_i = 1$ *so* $\text{order}(D_i) = 0$.*) Since* $\text{order}(W^{-1}) = \text{order}(W_1) + \text{order}(W_2)$ *in the generic case, we conclude that the generic order of the mixed sensitivity problem is lower than that in the abstract problem considered in [50] by* $2\ \text{order}(W_2)$.

Let

$$\begin{aligned} c_j &= [(\Pi_+ A - \Pi_- B)x_\mu](w_j) \\ &= (\Pi_+ A x_\mu)(w_j) - (\Pi_- B x_\mu)(w_j), j = 1, ..., k, \end{aligned}$$

$c_{k+1}, \ldots, c_{2k}, c_{2k+1}, \ldots, c_{2k+m}$,

$$c_{2k+m+j} = E(\bar{d}_j) R^{D_i^*}(\bar{d}_j) x_\mu(\bar{d}_j), j = 1, ..., n, \tag{B.3}$$

$$c_{2k+m+n+j} = E^* R^{D_i}[\Pi_-(A+B)x_\mu](-d_j), j = 1, ..., n. \tag{B.4}$$

From (B.1) we have

$$\begin{aligned} (\mu^2 - W_1^* W_1)x_\mu &= \Delta_\mu x_\mu \\ &= \sum_{j=1}^{2k+m+2n} c_j \phi_j. \end{aligned} \tag{B.5}$$

In the following, we shall show how to get a system of $2k + m + 2n$ homogeneous linear equations in $c_1, \ldots, c_{2k+m+2n}$.

1. Solve

   $$\mu^2 - W_1^* W_1 = 0$$

   to get $2k$ roots $r_1, \ldots r_{2k}$. So

   $$\sum_{j=1}^{2k+m+2n} c_j \phi_j(r_l) = 0, l = 1, ..., 2k.$$

2. By Proposition 2, $R^{x_\mu}(p_l^{B_W}) = 0, l = 1, ..., m$. So we have

   $$\begin{aligned} 0 &= \lim_{s \to p_l^{B_W}} (p_l^{B_W} - s)(\mu^2 - W_1^* W_1)x_\mu \\ &= \lim_{s \to p_l^{B_W}} (p_l^{B_W} - s)\Delta_\mu x_\mu, \quad \text{for} \quad l = 1, ..., m, \end{aligned}$$

   which are

   $$\sum_{j=1}^{k} \frac{c_{k+j}}{(w_1 + p_l^{B_W})} - \sum_{j=1}^{m} \frac{c_{2k+j}}{(p_j^{B_\mu} - p_l^{B_W})} - \sum_{j=1}^{n} \frac{c_{2k+m+j}}{(\bar{d}_j - p_l^{B_W})} = 0 \quad \text{for} \quad l = 1, \ldots, m.$$

3.

$$[(\mu^2 - W_1^* W_1)x_\mu](\bar{d}_l) = \sum_{j=1}^{2k+m+2n} c_j \phi_j(\bar{d}_l), \quad \text{for} \quad l = 1, ..., n.$$

4.

$$\begin{aligned}
[(\mu^2 - W_1^* W_1)R^{x_\mu}](-d_l) &= \lim_{s \to -d_l} (d_l + s)(\mu^2 - W_1^* W_1)x_\mu \\
&= \lim_{s \to -d_l} (d_l + s)(\Delta_\mu x_\mu) \\
&= (\lim_{s \to -d_l} (d_l + s)A^*) \left(\sum_{j=1}^{k} \frac{c_{k+j}}{w_j - d_l} - \sum_{j=1}^{m} \frac{c_{2k+j}}{p_j^{B_\mu} + d_l} \right. \\
&\quad \left. - \sum_{j=1}^{n} \frac{c_{2k+m+j}}{\bar{d}_j + d_l} \right) + c_{2k+m+n+l}, \quad \text{for} \quad l = 1, ..., n.
\end{aligned}$$

From (B.3) we get

$$x_\mu(\bar{d}_l) = \frac{c_{2k+m+l}}{E(\bar{d}_l) R^{D_i^*}(\bar{d}_l)}, \quad \text{for} \quad l = 1, ..., n.$$

Since $x_\mu \in B_\mu B_W N_i D_i \mathbf{H}^2_-$, we see that

$$Ax_\mu(-d_l) = (\frac{W_1^* W_1}{W^*} N_i \frac{R^{x_\mu}}{R^{D_i}})(-d_l), \quad \text{for} \quad l = 1, ..., n.$$

So

$$\begin{aligned}
(\Pi_- Ax_\mu)(-d_l) &= (Ax_\mu)(-d_l) - (\Pi_+ Ax_\mu)(-d_l) \\
&= (\frac{W_1^* W_1}{W^*} N_i \frac{R^{x_\mu}}{R^{D_i}})(-d_l) - \sum_{j=1}^{k} \frac{c_{k+j}}{w_j - d_l} + \sum_{j=1}^{m} \frac{c_{2k+j}}{p_j^{B_\mu} + d_l}, \quad \text{for} \quad l = 1, ..., n.
\end{aligned}$$

Also

$$(\Pi_- Bx_\mu)(-d_l) = -\sum_{j=1}^{n} \frac{c_{2k+m+j}}{\bar{d}_j + d_l}, \quad \text{for} \quad l = 1, ..., n.$$

From (B.4) we have

$$\begin{aligned}
R^{x_\mu}(-d_l) &= (\frac{W^*}{W_1^* W_1 E^* N_i^*})(-d_l)[c_{2k+m+n+l} + (E^* R^{D_i})(-d_l)(\sum_{j=1}^{k} \frac{c_{k+j}}{w_j - d_l} - \sum_{j=1}^{m} \frac{c_{2k+j}}{p_j^{B_\mu} + d_l} + \\
&\quad + \sum_{j=1}^{n} \frac{c_{2k+m+j}}{\bar{d}_j + d_l})], \quad \text{for} \quad l = 1, ..., n.
\end{aligned}$$

Summarizing the results from above, we have the following procedure to find eigenvalues:

1. Solve

$$\mu^2 - W_1^* W_1 = 0$$

to get $2k$ roots $r_1, ..., r_{2k}$.

Let

$$a_{ij} = \phi_j(r_i), \quad \text{for} \quad i = 1, ..., 2k, j = 1, ..., 2k + m + 2n.$$

2. For $i = 1, ..., m$, let

$$\begin{aligned}
a_{2k+i,k+j} &= \frac{1}{w_j + p_i^{B_W}}, \quad \text{for} \quad j = 1, ..., k; \\
a_{2k+i,2k+j} &= -\frac{1}{p_j^{B_\mu} - p_i^{B_W}}, \quad \text{for} \quad j = 1, ..., m; \\
a_{2k+i,2k+m+j} &= -\frac{1}{\bar{d}_j - p_i^{B_W}}, \quad \text{for} \quad j = 1, ..., n.
\end{aligned}$$

3. For $i = 1, ..., n$, let

$$\begin{aligned}
a_{2k+m+i,j} &= \phi_j(\bar{d}_i), \quad \text{for} \quad j = 1, ..., 2k + m; \\
a_{2k+m+i,2k+m+j} &= \begin{cases} \phi_{2k+m+j}(\bar{d}_i) - \frac{(\mu^2 - W_1^* W_1)}{E R^{D_i^*}}(\bar{d}_j), & \text{for} \quad i = j, j = 1, ..., n, \\ \phi_{2k+m+j}(\bar{d}_i), & \text{for} \quad i \neq j, j = 1, ..., n, \end{cases} \\
a_{2k+m+i,2k+m+n+j} &= \phi_{2k+m+n+j}(\bar{d}_i), \quad \text{for} \quad j = 1, ..., n,
\end{aligned}$$

where $E = -W N_o X$.

4. For $i = 1, ..., n$, let

$$\begin{aligned}
a_{2k+m+n+i,k+j} &= \{[\frac{(\mu^2 - W_1^* W_1)W^*}{W_1^* W_1 N_i^*} R^{D_i}](-d_i) - R^{A^*}(-d_i)\}\frac{1}{w_j - d_i}, \\
&\quad \text{for} \quad j = 1, ..., k; \\
a_{2k+m+n+i,2k+j} &= -\{[\frac{(\mu^2 - W_1^* W_1)W^*}{W_1^* W_1 N_i^*} R^{D_i}](-d_i) - R^{A^*}(-d_i)\}\frac{1}{d_i + p_j^{B_\mu}}, \\
&\quad \text{for} \quad j = 1, ..., m; \\
a_{2k+m+n+i,2k+m+j} &= \{[\frac{(\mu^2 - W_1^* W_1)W^*}{W_1^* W_1 N_i^*} R^{D_i}](-d_i) + R^{A^*}(-d_i)\}\frac{1}{d_i + \bar{d}_i}, \\
&\quad \text{for} \quad j = 1, ..., n; \\
a_{2k+m+n+i,2k+m+n+j} &= \begin{cases} [\frac{(\mu^2 - W_1^* W_1)W^*}{W_1^* W_1 E^* N_i^*}](-d_i) - 1, & \text{for} \quad i = j, j = 1, ..., n; \\ 0, & \text{for} \quad i \neq j, j = 1, ..., n. \end{cases}
\end{aligned}$$

5. Let all other $a_{ij} = 0$ for $1 \leq i, j \leq 2k + m + 2n$.

We form the matrix

$\mathcal{A}(\mu) =$

$$\begin{pmatrix}
a_{11} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1,2k+m+2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{2k,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{2k,2k+m+2n} \\
0 & \cdots & 0 & a_{2k+1,k+1} & \cdots & a_{2k+1,2k+m+n} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & a_{2k+m,k+1} & \cdots & a_{2k+m,2k+m+n} & 0 & \cdots & 0 \\
a_{2k+m+1,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{2k+m+1,2k+m+2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{2k+m+n,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{2k+m+n,2k+m+2n} \\
0 & \cdots & 0 & a_{2k+m+n+1,k+1} & \cdots & a_{2k+m+n1,2k+m+n} & a_{2k+m+n+1,2k+m+n+1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & a_{2k+m+2n,k+1} & \cdots & a_{2k+m+2n,2k+m+n} & 0 & \cdots & a_{2k+m+2n,2k+m+2n}
\end{pmatrix}$$

and we can then write

$$\mathcal{A}(\mu)C = 0, \tag{B.6}$$

where $C = (c_1, \ldots, c_{2k+m+2n})^T$.

In order for (B.6) to have nontrivial solutions, we set

$$\det[\mathcal{A}(\mu)] = 0, \tag{B.7}$$

and solve (B.7) for μ^2 to be an eigenvalue.

Appendix C

Proof of Lemma 2

Define

$$\bar{X} \triangleq W - M\bar{Q} \ .$$

Let

$$X_n(j\omega) \triangleq W(j\omega) - M(j\omega)T_n(j\omega).$$

We now consider the magnitude squared of $X_n(j\omega)$,

$$|X_n(j\omega)|^2 = |W(j\omega) - M(j\omega)T_n(j\omega)|^2 = |W(j\omega) + h_n(j\omega)[\mu_i e^{i\alpha(\omega)} - W(j\omega)]|^2$$

using the definitions in (3.20). Noting the assumption (3.13) we take

$$W_\infty \triangleq \sup \ R_W(\infty),$$

and so for any $\nu > 0$ we can take $n(\nu)$ large enough so that

$$|W(j\omega)| < W_\infty + \nu \qquad a.e. \text{ for } |\omega| > n(\nu). \tag{C.1}$$

Now we fix ν, assume that n satisfies (C.1), and later determine how to pick ν as a function of ϵ to satisfy

$$\|X_n\| \leq \mu_i + \epsilon. \tag{C.2}$$

Suppose ω_n is a frequency (possibly ∞) at which $\|X_n\| \in R_{X_n}(\omega_n)$. If ω_n is finite, define

$$W_n = \sup \ R_W(\omega_n),\ h = |h_n(j\omega_n)|,\ \delta = \arg[h_n(j\omega_n)] \quad \text{and} \quad \alpha = \alpha(\omega_n).$$

Otherwise we know that $\|X_n\| = \mu_i$, and we define

$$W_n \triangleq \mu_i,\ h \triangleq 0,\ \delta \triangleq 0 \text{ and } \quad \alpha \triangleq 0.$$

Note that these are functions of n, as is ω_n. We also note for later use that δ satisfies $0 \leq \delta \leq 2\pi/n$.

Now we show that the norm of X_n approaches the infimal value μ_i as n increases. For $\epsilon > 0$ let $B_\epsilon(\omega_n) = (\omega_n - \epsilon, \omega_n + \epsilon)$. Then for all ω

$$
\begin{aligned}
|X_n(j\omega)|^2 &\le \lim_{\epsilon\to 0} \operatorname*{ess\,sup}_{\zeta \in N_\epsilon(\omega_n)} |W(j\zeta) + h_n(j\zeta)(\bar{X}(j\zeta) - W(j\zeta))|^2 \\
&= \lim_{\epsilon\to 0} \operatorname*{ess\,sup}_{\zeta \in N_\epsilon(\omega_n)} \Big(|W(i\zeta)|^2 + 2Re\left\{W^*(i\zeta)h_n(i\zeta)(\bar{X}(i\zeta) - W(i\zeta))\right\} + \\
&\qquad |h_n(i\zeta)(\bar{X}(i\zeta) - W(i\zeta))|^2 \,\Big) \\
&= \lim_{\epsilon\to 0} \operatorname*{ess\,sup}_{\zeta \in N_\epsilon(\omega_n)} \Big(|W(i\zeta)|^2 - 2Re\{h_n(i\zeta)\}|W(i\zeta)|^2 + \\
&\qquad 2Re\left\{W^*(i\zeta)h_n(i\zeta)\bar{X}(i\zeta)\right\} + |h_n(i\zeta)|^2|\bar{X}(i\zeta) - W(i\zeta)|^2 \,\Big) \\
&= \lim_{\epsilon\to 0} \operatorname*{ess\,sup}_{\zeta \in N_\epsilon(\omega_n)} \Big(|W(i\zeta)|^2 - 2Re\{h_n(i\zeta)\}|W(i\zeta)|^2 + |h_n(i\zeta)|^2|W(i\zeta)|^2 \\
&\qquad + |h_n(i\zeta)|^2|\bar{X}(i\zeta)|^2 + 2Re\left\{W^*(i\zeta)h_n(i\zeta)\bar{X}(i\zeta)\right\} - \\
&\qquad 2|h_n(i\zeta)|^2 Re\left\{\bar{X}(i\zeta)W^*(i\zeta)\right\} \Big).
\end{aligned}
$$

For $n \ge 4$, $0 \le \arg(h_n(i\zeta)) \le \pi/2$, and we write

$$
\begin{aligned}
|X_n(j\omega)|^2 &\le \lim_{\epsilon\to 0} \operatorname*{ess\,sup}_{\zeta \in N_\epsilon(\omega_n)} \Big(|W(i\zeta)|^2 - 2|h_n(i\zeta)|\cdot|W(i\zeta)|^2 + |h_n(i\zeta)|^2|W(i\zeta)|^2 + \\
&\qquad 2(|h_n(i\zeta)| - Re\{h_n(i\zeta)\})|W(i\zeta)|^2 + |h_n(i\zeta)|^2|\bar{X}(i\zeta)|^2 + \\
&\qquad 2Re\{h_n(i\zeta)\}\, Re\left\{W^*(i\zeta)\bar{X}(i\zeta)\right\} - 2Im\{h_n(i\zeta)\}\, Im\left\{W^*(i\zeta)\bar{X}(i\zeta)\right\} - \\
&\qquad 2|h_n(i\zeta)|^2 Re\left\{\bar{X}(i\zeta)W^*(i\zeta)\right\} \Big) \\
&= \lim_{\epsilon\to 0} \operatorname*{ess\,sup}_{\zeta \in N_\epsilon(\omega_n)} \Big(|W(i\zeta)|^2 \left[(1 - |h_n(i\zeta)|)^2 + 2(|h_n(i\zeta)| - Re\{h_n(i\zeta)\})\right] \\
&\qquad + |h_n(i\zeta)|^2|\bar{X}(i\zeta)|^2 + 2\Big(Re\left\{W^*(i\zeta)\bar{X}(i\zeta)\right\}[Re\{h_n(i\zeta)\} - \\
&\qquad |h_n(i\zeta)|^2\,] - Im\{h_n(i\zeta)\}\, Im\left\{W^*(i\zeta)\bar{X}(i\zeta)\right\} \Big)
\end{aligned}
$$

Using continuity of h_n and the definition of W_n we get

$$
\begin{aligned}
|X_n(j\omega)|^2 &\le W_n^2 \cdot \left[(1-h)^2 + 2h(1 - cos\delta)\right] + h^2\mu_i^2 + 2W_n\mu_i\{h(1-h) + hsin\delta\} \\
&\le [W_n(1-h) + h\mu_i]^2 + 2h\delta^2 W_n^2 + 2W_n\mu_i h\delta
\end{aligned}
$$

Given n, there are two possibilities:
case (i). ($\omega_n > n$) In this case we shall use the fact that $W_n \to W_\infty$ as $\omega_n \to \infty$.

$$
|X_n(j\omega)|^2 \le [(W_\infty + \nu)(1-h) + h\mu_i]^2 + 2h\delta^2(W_\infty + \nu)^2 + 2h\mu_i\delta(W_\infty + \nu)
$$

$$
\begin{aligned}
&\leq [(\mu_i+\nu)(1-h)+h\mu_i]^2+2h\delta^2(\mu_i+\nu)^2+2h\mu_i\delta(\mu_i+\nu)\\
&\leq [\mu_i+\nu]^2+2\mu_i\delta(\mu_i+\nu)+2\delta^2(\mu_i+\nu)^2\\
&= \mu_i^2+\nu\cdot 2\mu_i(1+\delta)+2\mu_i^2\delta+\nu^2+2\delta^2(\mu_i+\nu)^2\\
&\leq \mu_i^2+\nu\cdot 2\mu_i(1+\frac{2\pi}{n})+4\pi\mu_i^2/n+\nu^2+2\left(\frac{2\pi}{n}\right)^2(\mu_i+\nu)^2.
\end{aligned}
$$

Now we simply state how to pick n_ϵ and $\nu(\epsilon)$ such that $n \geq n_\epsilon$ and $0 \leq \nu \leq \nu(\epsilon)$ guarantees that the right hand side of the last inequality gives

$$|X_n(j\omega)|^2 \leq \mu_i^2+\epsilon.$$

Assuming $0 < \epsilon \leq 1$, we take

$$0 \leq \nu(\epsilon) \leq \frac{\epsilon}{16\mu_i}, \text{ and } \quad n_\epsilon \geq 2\pi \text{ such that } |W(j\omega)| < |W_\infty| + \frac{\epsilon}{16\mu_i} \text{ for } |\omega| > n_\epsilon, \tag{C.3}$$

which gives $\nu \cdot 2\mu_i(1+\frac{2\pi}{n}) \leq \epsilon/4$. Taking

$$n_\epsilon > \frac{16\pi\mu_i^2}{\epsilon} \tag{C.4}$$

gives $\pi\mu_i^2/n \leq \epsilon/4$ for $n \geq n_\epsilon$. Taking

$$\nu(\epsilon) \leq \frac{\epsilon}{2} \tag{C.5}$$

gives $\nu^2 \leq \epsilon/4$. Finally, using (22) with $0 \leq \nu \leq \nu(\epsilon)$ and taking

$$n_\epsilon \geq \frac{8\pi(\mu_i+1)}{\epsilon} \tag{C.6}$$

allows us to conclude that $8\left(\frac{\pi}{n}\right)^2(\mu_i+\nu)^2 \leq \epsilon/4$,

Thus (C.3) — (C.6) together will ensure $\|X_n\|_\infty^2 \leq \mu_i^2+\epsilon$ in this case.

case (ii). ($\omega_n \leq n$) The idea in this case is that $h \to 1$ since $\omega_n \leq n$. Since $\|W\|_\infty \geq |W_n|$ and $h \leq 1$,

$$
\begin{aligned}
|X_n(j\omega_n)|^2 &\leq [W_\infty(1-h)+h\mu_i]^2+2h\delta^2W_\infty^2+2W_\infty\mu_i h\delta\\
&\leq \mu_i^2+(1-h)\cdot 2\|W\|_\infty\mu_i+\|W\|_\infty^2(1-h)^2+4\pi\mu_i\|W\|_\infty/n\\
&\qquad +2\left(\frac{2\pi}{n}\right)^2\|W\|_\infty^2.
\end{aligned} \tag{C.7}
$$

We proceed to satisfy (C.2) by bounding each of the last four terms of the right hand side of (C.7) to be individually bounded by $\epsilon/4$. Now since $\omega_n \leq n$ in this case,

$$
\begin{aligned}
1 &\geq h^2 = |h(j\omega_n)|^2\\
&= [\alpha^2/(\gamma^2+\omega_n^2)]^{1/n}\\
&\geq [\alpha^2/(\gamma^2+n^2)]^{1/n}
\end{aligned}
$$

It is easy to check that $\lim_{n\to\infty}[\alpha^2/(\gamma^2+n^2)]^{1/n}=1$, so $h^2 \to 1$ as $n \to \infty$, and therefore $(1-h) \to 0$. In order to estimate the required n_ϵ, we want $(1-h)\cdot 2\|W\|\mu_i < \epsilon/4$. It is sufficient to take

$$1-\frac{\epsilon}{8\|W\|\mu_i} < [\alpha^2/(\gamma^2+n^2)]^{1/2n},$$

or

$$\log(1-\frac{\epsilon}{8\|W\|\mu_i}) < \frac{1}{2n}\log[\alpha^2/(\gamma^2+n^2)].$$

For $n \geq \gamma$

$$\alpha^2/(\gamma^2+n^2) \geq \alpha^2/2n^2,$$

so a sufficient condition is

$$\log(1-\frac{\epsilon}{8\|W\|\mu_i}) < \frac{2\log(\alpha)-2\log(n)-\log(2)}{2n}.$$

Assuming $n \geq \max(2, 1/\alpha^2)$, a sufficient condition is then

$$\log(1-\frac{\epsilon}{8\|W\|\mu_i}) < -\frac{2\log(n)}{n}. \tag{C.8}$$

Similarly, $\|W\|_\infty^2(1-h)^2 \leq \epsilon/4$ can be assured by requiring

$$log(1-\frac{\sqrt{\epsilon}}{2\|W\|}) < -\frac{2\log(n)}{n}. \tag{C.9}$$

$4\pi\mu_i\|W\|_\infty/n \leq \epsilon/4$ will be obtained by assuming

$$n_\epsilon \geq \frac{16\pi\mu_i\|W\|}{\epsilon}$$

and $2\left(\frac{2\pi}{n}\right)^2\|W\|_\infty^2 \leq \epsilon/4$ by

$$n_\epsilon > 8\pi\|W\|\sqrt{\frac{2}{\epsilon}}\ . \tag{C.10}$$

Thus using (C.3)-(C.10) we pick n_ϵ to satisfy:

$$\begin{cases} n_\epsilon > \max\left(2\pi,\ \frac{8\pi\mu_i^2}{\epsilon},\ \frac{1}{\alpha^2},\ \frac{16\pi\mu_i\|W\|}{\epsilon},\ 8\pi\|W\|\sqrt{2/\epsilon}\right) \\ |W(j\omega)| < |W_\infty| + \frac{\epsilon}{8\mu_i} \qquad \text{for } |\omega| > n_\epsilon \\ \frac{-2\log(n_\epsilon)}{n_\epsilon} > \max\left(\log(1-\frac{\epsilon}{8\|W\|\mu_i}),\ \log(1-\frac{\sqrt{\epsilon}}{2\|W\|})\right) \end{cases}$$

to obtain $\|X_n\|^2 - \mu_i^2 \leq \epsilon$. ∎

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